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Multiquantum systems and point processes I.
Generating functionals and nonlinear semigroups

V.P. Belavkin*

School of Theoretical Physics
Dublin Institute for Advanced Studies
10 Burlington Road, Dublin 4, Ireland

Abstract.

An algebraic approach to representation theory and the description of multicomponent quantum systems is considered. A generating multiquantum state functional and nonlinear completely positive map are introduced and a dilation theorem giving a nonlinear extension of GNS and Stinenspring theorem is proved. A number particle operator-valued weight and an empirical weight operator generating a macroscopic inductive algebra are defined, and asymptotic commutativity of this algebra is proved. A canonical multiquantum stochastic process called quasi-Poissonian is constructed and the general structure of the generator for infinite divisible multi-quantum states as well as multiquantum semigroups is found. An existence theorem extending the Lindblad theorem to unbounded generators as well as nonlinear generators is proven. The class of quasifree quantum point stochastic processes is introduced to describe Markovian dynamics of non-interacting quantum particles and corresponding birth, branching and current nonlinear semigroups and their generators are studied.

* On leave of absence from M.I.E.M., B. Vusovski Street 3/12,
Moscow 109028, USSR.

Contents

Introduction.

This paper is written on the basis of lectures given by the author in the Dublin Institute for Advanced Studies in September 1987. It contains a mathematical introduction to the quantum algebraic theory of multicomponent systems which are the systems of a random number of quantum particles, or quasiparticles, called shortly quantums, of the same or different types. Such multiquantum systems are described by a decomposable algebra of observables compatible with the total number operator on Fock space, containing both the Bose and Fermi states.

In the first chapter we consider questions of representation and composition of such systems arising in large number theory and large deviations. Then we develop an analytical apparatus of state generating functionals and nonlinear completely positive maps. We prove a theorem giving the necessary and sufficient conditions for an analytical functional and map to be the generator for a multiquantum state and linear completely positive map. This theorem gives a nonlinear extension of GNS construction for states and Stinenspring-Kraus construction for maps [1].

Then we introduce a notion of number weight operator, which is a C*-algebraic generalization of integer valued random measure, describing the distribution of particles in phase space, and a notion of emprical weight operator, which is the noncommutative analogue of empirical measure, describing the distribution of particles in the composition of a number N of identical systems, divided by N. We prove that the macroscopic algebra generated by the empirical weight operator is an asymptotically Abelian algebra in an inductive limit.

The second chapter is concerned with models and properties of quantum stochastic point processes, which are quantum stochastic processes over the multiquantum algebra. Firstly we introduce a quasi-Poissonian quantum process, generating the simplest class of Markovian multiquantum stochastic processes that are quantum birth Poissonian processes. We describe such a process by a multiplicative semigroup of analytical generating functionals, and find the necessary and sufficient conditions for an analytical functional to be a generator of infinitely divisible multiquantum state. This is a particular case of a proven general theorem giving a representation of conditionally completely positive maps with respect to an analytical representation of multiplicative

*-semigroup. In this way we obtain a nonlinear extension of the Evans-Lewis construction for a generator of a linear completely positive semigroup [2]. Then we prove the existence theorem for multiquantum Markovian master equations, which generalizes the Lindblad construction [3] of Markovian semigroup on a class of unbounded generators.

This gives us an analytical tool for the mathematical description of quantum Markovian processes for interacting particles in Fock space, having unbounded generators. The simplest such multiquantum process is the branching quantum process, which is described by an unbounded sum of single-quantum branching generators. We show that this process is generated by a nonlinear backward master equation for one-particle observables, and prove the existence theorem for corresponding nonlinear semigroups. This gives us a possibility to construct the current semigroups to quasi-free quantum point processes, described in the general case by a completely dissipative pair of current semialgebras.

The second part of this quantum theory of multicomponent systems and point processes will contain the large number theorems and large deviations, which we used in [4-6] for the disclosure of the general structure of quantum kinetic equations for weakly interacting particles. The number of particles which collide may be arbitrary and not equal to the number of particles which appear as the result of the collision, i.e. we can have the processes of birth, death, splitting and collapse, with mutual transformation of particles of one type into particles of another type. It is evident that such complicated processes cannot be described within the frames of Hamiltonian formalism [4] in a given Fock space of a variable number of particles of the considered types, since this formalism demands that the considered system should be closed, and in particular, the number of particles should be conserved. Nevertheless, due to the condition of complete positiveness for the corresponding Markovian transition mappings we can consider these processes as quantum random processes in the weak sense of [7], or even in the strong sense [8], if we extend the system under consideration to a "closed large system" by using some dilation construction. We will show that in this non-Hamiltonian situation the infinitesimal description of multicomponent system is asymptotically reduced to the canonical pair of typically irreversible one-particle nonlinear equations - forward for average weight operator and backward for single-particle observables. The first one containing a Vlasov as well as Boltzmann kinetic type equation corresponds to the large number limit, and the second one corresponds to the large deviation limit. We will prove that the theorem announced in [6], this canonical pair, firstly derived in [5], is the sytems of Bellmann-Pontrjagin equations for the variational problem, corresponding to Ventzel-Frejdlin large deviations in the phase space $\mathfrak{A}^*_+$, which is the positive cone of a single-particle C*-algebra $\mathfrak{A}$.

Chapter 1.

An algebraic approach to quantum multicomponent systems

1. The algebra of multiquantum observables and multicomponent quantum systems

Let $\mathcal{A}$ be a W*-algebra, called the single-body, or one-quantum algebra for a quantum many-particle quantum-point system.

One can assume that $\mathcal{A}$ is represented as a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{E})$ on a Hilbert space $\mathcal{E}$, where $\mathcal{B}(\mathcal{E})$ denotes the algebra of all bounded operators on $\mathcal{E}$. For example in the case of pure quantum particles of several types $i \in J$ $\mathcal{A}$ is the algebra of decomposable operators $A = \oplus A_i$, $A_i \in \mathcal{B}(\mathcal{E}_i)$ admitting the vector states $\xi = \oplus \xi_i$, $\xi_i \in \mathcal{E}_i$ of a particle with random type $i \in J$. Such $\mathcal{A}$ are satisfying the compatibility (commutativity) condition of the observables $A \in \mathcal{A}$ on the Hilbert sum $\mathcal{E} = \oplus \mathcal{E}_i$ of one-particle states $\xi \in \mathcal{E}$, with the observable of particle type, which is described by the identity resolution $I = \oplus I_i$ (by type-index operator $\hat{\imath} = \oplus i I_i$ if $i \in \mathbb{N}$), where I_i is the orthoprojector on the sub-space $\mathcal{E}_i$ of particle states with type i.

We denote by $\mathcal{A}_*$ the predual space of $\mathcal{A}$, which is the space of decomposable trace-class operators $\rho = \oplus \rho^{(i)}$, if $\mathcal{A} = \oplus \mathcal{B}(\mathcal{E}_i)$, and by $\langle \rho, A \rangle$ the pairing of $A \in \mathcal{A}$ and $\rho \in \mathcal{A}_*$, inducing the w*-topology on $\mathcal{A}$, which is $\langle \rho, A \rangle = \sum Tr\, \rho^{(i)} A_i$, if $A = \oplus A_i$.

The algebra of n-quantum (n-body) observables is the symmetrical n-th tensor W*-power $\mathcal{A}^{(n)}$, which can be defined as the von Neumann algebra, generated on Hilbert tensor power $\mathcal{E}^{\otimes n}$ by the operators $A^{\otimes n}$ for all $A \in \mathcal{A}$. A linear w*-continuous functional $\phi_n : A^{(n)} \in \mathcal{A}^{(n)} \mapsto \langle \phi_n, A^{(n)} \rangle$ is called n-quantum state, if it is positive $\phi_n \geq 0$, or $\langle \phi_n, A^{(n)} \rangle \geq 0$, if $A^{(n)} \geq 0$, and normalized:

$$\| \phi_n \|_* = \langle \phi_n, I^{\otimes n} \rangle = 1 \quad , \quad or \quad Tr\, \phi_n = 1 \quad , \quad if \quad \mathcal{A} = \mathcal{B}(\mathcal{E}).$$

In the case $\mathcal{A} = \oplus \mathcal{B}(\mathcal{E}_i)$, $\mathcal{A}^{(n)}$ is a decomposable W*-algebra:

$A^{(n)} = \bigoplus_{i_1 \in J} \cdots \bigoplus_{i_n \in J} A^{(n)}_{i_1 \ldots i_n}$, where $A^{(n)}_{i_1 \ldots i_n}$ are symmetrical bounded operators on $\mathcal{E}_{i_1} \otimes \ldots \otimes \mathcal{E}_{i_n}$ and

$$\langle \varphi_n , A^{(n)} \rangle = \sum_{i_1 \in J} \ldots \sum_{i_n \in J} Tr\ \varphi_n^{(i_1, \ldots, i_n)} A^{(n)}_{i_1 \ldots i_n} ,$$

where $\varphi_n^{(i_1 \ldots i_n)}$ are symmetrical trace-class operators.

The multiquantum (many-body) system over $\mathcal{A}$ with random number of particles n = 0,1,2, ... is described by the W*-algebra $\mathcal{M} = \bigoplus_{n<\infty} \mathcal{A}^{(n)}$ of the observables $\hat{A} = \oplus A^{(n)}$, which commute with particle number operator $\hat{n} = \oplus\, n\, I^{\otimes n}$ on Fock space $\mathcal{F} = \bigoplus_{n<\infty} \mathcal{E}^{\otimes n}$ ($\mathcal{A}^{(0)} = \mathbb{C}$ as usual). A linear w*-continuous functional $\varphi \in \mathcal{M}_*$ has the form $\langle \varphi , \hat{A} \rangle = \sum_{n=0}^{\infty} \langle \varphi_n , A^{(n)} \rangle$, and is called multiquantum state, if it is positive: $\varphi \geqslant 0$ and $\langle \varphi , I^{\otimes} \rangle \leqslant 1$, where $I^{\otimes} = \oplus I^{\otimes n}$. If $\varphi = [\varphi_n]$ is normalized: $\langle \varphi , I^{\otimes} \rangle = \Sigma \langle \varphi_n , I^{\otimes n} \rangle = 1$ which corresponds to the state with random finite number of particles n = 0,1,2,... with probability distribution $p_n = \langle \varphi_n , I^{\otimes n} \rangle$, otherwise the difference $1 - \Sigma p_n$ is considered as the probability p_∞ of infinite n.

The following theorem shows that the given construction of multiquantum system defines a W*-algebraic functor $m : \mathcal{A} \mapsto \mathcal{M}$ into the category of algebras $\mathcal{M} = \oplus \mathcal{A}^{(n)}$, which constructs the multiquantum system over a sum decomposable W*-algebra $\mathcal{B} = \oplus \mathcal{A}_i$ as the multiple composition $m(\mathcal{B}) = \otimes \mathcal{M}_i$ of multiquantum systems $\mathcal{M}_i = m(\mathcal{A}_i)$ over $\mathcal{A}_i \subseteq \mathcal{B}(\mathcal{E}_i)$ with different types $i \in J$.

Theorem I.1. Let $\lambda : \mathcal{A} \to \mathcal{B}$ be a W*-homomorphism of $\mathcal{A}$ into a W*-algebra $\mathcal{B}$ on a Hilbert space $\mathcal{G}$. Then the map $\lambda^{\otimes} = \oplus \lambda^{\otimes n}$ defined on $X^{\otimes} = \oplus X^{\otimes n}$ as $\lambda^{\otimes}(X^{\otimes}) = \lambda(X)^{\otimes}$ for all $X \in \mathcal{A}$, $X^+ X \leqslant I$, can be uniquely extended to a W*-homomorphism $m(\lambda)$ of multiquantum W*-algebra $m(\mathcal{A}) = \bigoplus_{n<\infty} \mathcal{A}^{(n)}$ into the algebra $m(\mathcal{B}) = \bigoplus_{n<\infty} \mathcal{B}^{(n)}$ on the Fock space $\mathcal{H} = \bigoplus_{n<\infty} \mathcal{G}^{\otimes n}$. If $\lambda = \lambda_1 \circ \lambda_2$, then $m(\lambda) = m(\lambda_1) \circ m(\lambda_2)$, and if $\mathcal{B}$ is a direct sum $\oplus \mathcal{A}_i$ of von Neumann algebras $\mathcal{A}_i \subseteq \mathcal{B}(\mathcal{E}_i)$, $i \in J$, then the W*-algebra $m(\mathcal{B})$ is W*-equivalent to the von Neumann tensor product $\otimes \mathcal{M}_i$ of W*-algebras $\mathcal{M}_i = m(\mathcal{A}_i)$ on the Fock tensor product $\otimes \mathcal{F}_i$ of the Hilbert spaces. $\mathcal{F}_i = \bigoplus_{n<\infty} \mathcal{E}_i^{\otimes n}$.

Proof. Defining for a n the *-homomorphism $\lambda^{\otimes n}$ on linear span of

$\{A^{\otimes n} \mid A \in \mathcal{A}\}$ by linearity and extending it on all $A^{(n)}$ by w*-continuity, we shall obtain a W*-homomorphism $m^{(n)}(\lambda) : \mathcal{A}^{(n)} \to \mathcal{B}^{(n)}$ and, hence, $m(\lambda) = \oplus m^{(n)}(\lambda)$ as a unique W*-homomorphism $m(\mathcal{A}) \to m(\mathcal{B})$, satisfying the condition $m(\lambda)(X^{\otimes}) = \lambda(X)^{\otimes}$, $X \in \{A \in \mathcal{A} \setminus A^{+}A \leq I\}$. Obviously, $m : \lambda \mapsto \lambda^{\otimes}$ is a multiplicative and identity-preserving map with respect to the composition $\lambda = \lambda_1 \circ \lambda_2$ and identity authomorphisms $\iota_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$. Remembering the definition of an (infinite) von Neumann product $\otimes \mathcal{M}_i$, we can regard it as von Neumann algebra, generated by the operators $B^{\otimes \nu} = \underset{i \in J}{\otimes} A_i^{\otimes n_i}$ for all $B \in \mathcal{B}$ and $\nu = (n_i)$, $n_i = 0, 1, \ldots$ with $|\nu| = \sum n_i < \infty$ and $A_i^{\otimes n_i} = 1$ for a $n_i = 0$ on corresponding subspaces $\mathcal{G}^{\otimes \nu} = \otimes \mathcal{E}_i^{\otimes n_i}$ with vacuum multiplier $\mathcal{E}_i^{\otimes n_i} = \mathbb{C}$, if $n_i = 0$. On the other hand the generating operators $B^{\otimes n} \overset{i}{=} (\oplus A_i)^{\otimes n}$ for $\mathcal{B}^{(n)}$ on $\mathcal{G}^{\otimes n} = (\oplus \mathcal{E}_i)^{\otimes n}$ can be decomposed up to the direct sum of unitary permutations of $(i_1, \ldots, i_n)$ as

$$B^{\otimes n} = \underset{i_1 \in J}{\oplus} \cdots \underset{i_n \in J}{\oplus} (A_{i_1} \otimes \cdots \otimes A_{i_n}) \simeq \underset{|\nu| = n}{\oplus} (B^{\otimes \nu} \otimes 1^{(n!/\nu!)}) \tag{1.1}$$

on $\mathcal{G}^{\otimes n} = \oplus \cdots \oplus (\mathcal{E}_{i_1} \otimes \cdots \otimes \mathcal{E}_{i_n}) \simeq \underset{|\nu| = n}{\oplus} (\mathcal{G}^{\otimes \nu} \otimes \mathbb{C}^{n!/\nu!})$,

where $\nu! = \prod_{i \in J} n_i!$ and $1^{(m)}$ is the identity operator in $\mathbb{C}^m$ with m being equal to the multiplicity of the product $B^{\otimes \nu}$ in the sum $B^{\otimes n}$. Hence the algebra $m(\oplus \mathcal{A}_i)$ is, up to unitary equivalence, the direct sum of von Neumann algebras $\mathcal{B}^{(\nu)} \otimes 1^{(n!/\nu!)}$ over all $|\nu| < \infty$, where $\mathcal{B}^{(\nu)} = \underset{i \in J}{\otimes} \mathcal{A}_i^{(n_i)}$ are the von Neumann tensor-product algebras defining the decomposition $\underset{i \in J}{\otimes} m(\mathcal{A}_i) = \underset{|\nu| < \infty}{\oplus} \mathcal{B}^{(\nu)}$. This proves the W*-equivalence $m(\oplus \mathcal{A}_i) \simeq \otimes m(\mathcal{A}_i)$ as the direct sums of the von Neumann algebras $\mathcal{B}^{(\nu)}$ and its multiplications $\mathcal{B}^{(\nu)} \otimes 1^{(|\nu|!/\nu!)}$ ▨

Corollary I.1. Let $\mathcal{B}$ be the w*-tensor product $\mathcal{A} \otimes \mathbb{C}^{(J)}$ of a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{E})$ and the abelian algebra $\mathbb{C}^{(J)}$ of bounded complex functions $i \mapsto c^{(i)}$ on J. Then $m(\mathcal{B})$ is w*-equivalent to the von Neumann tensor power $m(\mathcal{A})^{\otimes J}$ on Fock tensor power $\mathcal{F}^{\otimes J}$ of $\mathcal{F} = \overset{\infty}{\underset{n=0}{\oplus}} \mathcal{E}^{\otimes n}$ of cardinality $|J|$, and the W*-isomorphism $\iota(A) = A \otimes 1^{(J)}$ of $\mathcal{A}$ into $\mathcal{A} \otimes \mathbb{C}^{(J)}$ induces the W*-isomorphism $\hat{\iota} = \underset{|\nu| < \infty}{\oplus} \iota^{(\nu)}$ of $\mathcal{M} = m(\mathcal{A})$ into $\mathcal{M}^{\otimes J}$ on $\mathcal{F}^{\otimes J}$ defined by the imbeddings $\iota^{(\nu)}$ of $\mathcal{A}^{(|\nu|)}$ into $\mathcal{B}^{(\nu)} = \underset{i \in J}{\otimes} \mathcal{A}^{(n_i)}$, $|\nu| = \sum_{i \in I} n_i$. Representing $\mathcal{A} \otimes \mathbb{C}^{(J)}$ as direct sum $\mathcal{B} = \oplus \mathcal{A}_i$ of identical $\mathcal{A}_i = \mathcal{A}$, we obtain this corollary as a particular case of theorem 1 by the definition of the power $\mathcal{M}^{\otimes J}$ as the product $\otimes \mathcal{M}_i$ of identical $\mathcal{M}_i = \mathcal{M}$. The W*-isomorphism $\hat{\imath} : \mathcal{M} \to \mathcal{M}^{\otimes |J|}$, induced by the W*-isomorphism $m(\imath)$ and by W*-equivalence $m(\mathcal{A} \otimes \mathbb{C}^{(J)}) \simeq m(\mathcal{A})^{\otimes J}$ describes obviously the multiplication of the particles $\mathcal{A}$ by representation of the many body system $\mathcal{M}$ over $\mathcal{A}$ into the composition $\mathcal{M}^{\otimes J}$ of identical many body systems over $\mathcal{A}$ with cardinality number $|J|$.

The pre-dual map $\omega \in \mathcal{M}_*^{\otimes J} \mapsto \omega \circ \hat{\imath} \in \mathcal{M}_*$ describes the identification of these similar particles, differing only by type-index $i \in J$, as the composition of a multiquantum state ω of the composed particles system $\mathcal{M}^{\otimes J}$ and the multiplication isomorphism of the system $\mathcal{M}$ into $\mathcal{M}^{\otimes J}$. Taking into account that the n-quantum state on $\mathcal{B}^{(n)}$ over $\mathcal{B} = \oplus \mathcal{A}_i$ is decomposable as

$$\omega_n = \bigoplus_{i_1 \in J} \cdots \bigoplus_{i_n \in J} \omega_n^{(i_1 \ldots i_n)} = \bigoplus_{|\nu| = n} (\omega_\nu \otimes 1^{(n!/\nu!)}) \nu!/n! \, , \tag{1.2}$$

where $\omega_\nu \in \mathcal{B}_*^{(\nu)}$ is a state on $\mathcal{B}^{(\nu)} = \bigotimes_{i \in J} \mathcal{A}_i^{(n_i)}$ for a $\nu = (n_i)$, we identify the state $\varphi = \omega \circ \hat{\imath}$ on $\mathcal{M} = m(\mathcal{A})$ as the restriction of each component of $\omega = [\omega_n]$ on the W*-subalgebra $\mathcal{A}^{(n)} \otimes 1^{(J^n)}$ of $\mathcal{B}^{(n)} = \mathcal{A}^{(n)} \otimes \mathbb{C}^{(J^n)}$:

$$\langle \varphi_n, A^{(n)} \rangle = \sum_{i_1 \in J} \cdots \sum_{i_n \in J} \langle \omega_n^{(i_1 \ldots i_n)}, A^{(n)} \rangle = \sum_{|\nu| = n} \langle \omega_\nu, A^{(|\nu|)} \rangle . \tag{1.3}$$

Denoting $u^{(n)}_{j_1 \cdots j_n}$ the unitary isomorphism of the algebra $\mathcal{B}^{(\nu)}$ with $\nu = \delta^{j_1} + \ldots + \delta^{j_n} \equiv \mu(j_1, \ldots, j_n)$, i.e. with $n_i = m_i(j_1, \ldots, j_n)$,

$$m_i(j_1, \ldots, j_n) = \delta_i^{j_1} + \ldots + \delta_i^{j_n} \, , \qquad i \in J \tag{1.4}$$

$$\mu(j_1, \ldots, j_n) = (m_i(j_1, \ldots, j_n)) \qquad \text{for a } (j_1, \ldots j_n) \in J^n$$

onto the W*-algebra $\mathcal{A}^{(n)}_{j_1 \cdots j_n}$, generated by $A_{j_1} \otimes \ldots \otimes A_{j_n}$ for all $B = \oplus A_i$, such that

$$u^{(n)}_{j_1 \cdots j_n}(B^{\otimes \mu(j_1, \ldots, j_n)}) = A_{j_1} \otimes \ldots \otimes A_{j_n} \tag{1.5}$$

for a generating element $B^{\otimes\nu} = \underset{i\in J}{\otimes} A_i^{\otimes n_i}$ with $\nu = \mu(j_1,\dots,j_n)$, one can evaluate the state ω_ν as the composition

$$\omega_{\mu(j_1,\dots,j_n)} = \omega_n^{(j_1,\dots,j_n)} \circ u^{(n)}_{j_1\dots j_n} . \tag{1.6}$$

2. The state generating functionals and completely positive maps

Let $\mathcal{M} = m(\mathcal{A})$ be the W*-algebra of a multiquantum system over $\mathcal{A}$. We define the generating functional for a multiquantum state $\phi = [\phi_n]$ as a series

$$f(X) = \sum_{n=0}^{\infty} \langle \phi_n , X^{\otimes n} \rangle \equiv \langle \phi , X^{\otimes} \rangle, \tag{2.1}$$

which absolutely converges on the unit ball $\mathcal{A}^1 = \{X \in \mathcal{A} \mid X^*X \leq I\}$. If $\lambda : \mathcal{A} \to \mathcal{B}$ is a W*-homomorphism of $\mathcal{A}$ into a W*-algebra $\mathcal{B}$ with a multiquantum state $\omega : m(\mathcal{B}) \to \mathbb{C}$, then the composition $w \circ \lambda$ with the generating functional $w(Y) = \langle \omega , Y^{\otimes} \rangle$ is the generating functional $f(X) = w(\lambda(X))$ on $\mathcal{A}^1$ for the multiquantum state $\phi = \omega \circ m(\lambda)$, on the W*-algebra $\mathcal{M}$, represented in $\mathcal{N} = m(\mathcal{B})$. If $\mathcal{B} = \oplus \mathcal{A}_i$ is a decomposable algebra on the Hilbert sum $\mathcal{G} = \oplus \mathcal{E}_i$, then the generating functional $w[X_i] = w(\oplus X_i)$ is an analytical multifunctional on $X_i \in \mathcal{A}_i^1$, $i \in J$, which has a representation

$$w[X_i] = \sum_{|\nu| < \infty} \langle \omega_\nu , \underset{i\in J}{\otimes} X_i^{\otimes n_i} \rangle , \tag{2.2}$$

where ω_ν, $\nu = (n_i)$ are the corresponding states on the components $\mathcal{B}^{(\nu)} = \underset{i\in J}{\otimes} \mathcal{A}_i^{(n_i)}$ for $|\nu| = \sum_{i\in J} n_i < \infty$ of the von Neumann tensor representation $\underset{i\in J}{\otimes} m(\mathcal{A}_i) = \underset{|\nu|<\infty}{\oplus} \mathcal{B}^{(\nu)}$ of the W*-algebra $m(\oplus \mathcal{A}_i)$.

An analytical map $F : \mathcal{A}^1 \to \mathcal{B}$ is called w*-analytical in $\mathcal{A}^1$, if all all the derivatives

$$\delta_A^{(n)} F(X) = d^n F(X + tA)/dt^n \Big|_{t=0} \equiv F_A^{(n)}(X), \tag{2.3}$$

defined for a fixed X inside $\mathcal{A}^1$ as n-linear maps $A \in \mathcal{A} \mapsto F_A^{(n)}(X) \in \mathcal{B}$, have w*-continuous extensions on $\mathcal{A}^{(n)}$ i.e. if $F_A^{(n)}(0) = n! < \Phi_n, A^{\otimes n} >$, where Φ_n, $n = 0, 1, \ldots$ are linear w*-continuous $\mathcal{B}$-valued forms on $\mathcal{A}^{(n)}$.

It is obvious that the generating functional (2.1) is w*-analytical and satisfying the implications (iv) => (iii) => (ii) => (i) with $\mu(X) = \pi(X^{\otimes})$ for the GNS-representation $< \varphi, \hat{A} > = (\xi \,|\, \pi(\hat{A})\xi)$ of the state $\varphi : X^{\otimes} \mapsto f(X)$ in the following equivalent conditions.

Theorem I.2. Let F be a w*-continuous map $F : \mathcal{A}^1 \to \mathcal{B}$ into a W*-algebra $\mathcal{B}$ on Hilbert space $\mathcal{G}$, which is w*-analytical in the unit ball $\mathcal{A}^1$. Then the following conditions are equivalent:

(i) F is positive-definite (P.D.): for $M = 1, 2, \ldots$

$$\sum_{k,l=1}^{M} (\zeta^k \,|\, F(X_k^* X_l) \zeta^l) \geqslant 0 , \quad \forall \zeta^m \in \mathcal{G}, X_m \in \mathcal{A}^1, m \leq M .$$

(ii) F is completely positive i.e. $\hat{F} : \hat{X} \mapsto [F(X_l^k)]$ is positive:

$$(\hat{\zeta} \,|\, \hat{F}(\hat{X}^* \hat{X}) \hat{\zeta}) \geqslant 0, \quad \forall \hat{\zeta} \in \hat{\mathcal{G}}, \hat{X} \in \hat{\mathcal{A}}^1,$$

where $\hat{\mathcal{A}}^1$ is the unit ball of $\mathcal{A}$-valued matrices $\hat{X} = [X_l^k]$, $X_l^k \in \mathcal{A}$, $\hat{X}^* = [X_k^{l*}]$, acting in Hilbert space $\hat{\mathcal{E}} = \mathcal{E} \otimes \ell^2$ of sequences $\hat{\xi} = [\xi^m]$, $\xi^m \in \mathcal{E}$ as $\hat{X}\hat{\xi} = [\sum_i X_i^m \xi^i]$, and $\hat{F}(\hat{X})$ is the corresponding $\mathcal{B}$-valued matrix in $\hat{\mathcal{G}} = \mathcal{G} \otimes \ell^2$.

(iii) There exists a Hilbert space $\mathcal{K}$, an w*-analytical multiplicative *-homomorphism $\mu : \mathcal{A}^1 \to \mathcal{B}(\mathcal{K})$,

$$\mu(X^* Z) = \mu(X)^* \mu(Z) , \qquad \forall X, Z \in \mathcal{A}^1, \tag{2.4}$$

and an operator $J : \mathcal{G} \to \mathcal{K}$, commuting with commutant $\mathcal{B}' \ni Y$, $JY = \mu'(Y)J$ in a W*-representation $\mu' : \mathcal{B}' \to \mu(\mathcal{A}^1)'$, such that

$$F(X) = J^* \mu(X) J , \qquad \forall X \in \mathcal{A}^1. \tag{2.5}$$

(iv) There are linear w*-continuous completely positive $\mathcal{B}$-valued forms

$\Phi_n : A^{(n)} \mapsto \langle \Phi_n , A^{(n)} \rangle$ on $\mathcal{A}^{(n)}$ with $\sum_{n=0}^{\infty} \langle \Phi_n , I^{\otimes n} \rangle \in \mathcal{B}$, such that for all $X \in \mathcal{A}^1$

$$F(X) = \sum_{n=0}^{\infty} \langle \Phi_n , X^{\otimes n} \rangle \equiv \langle \Phi , X^{\otimes} \rangle . \tag{2.6}$$

Proof. Due to the obviousness of the (iii)$\Rightarrow$(ii)$\Rightarrow$(i) and Stinenspring theorem (iv)$\Rightarrow$(iii) [1] with $\mu(X) = \pi(X^{\otimes})$ constructed in [9] and μ' in [10], we have to prove only that (i) implies (iv) using for (2.3) the Cauchy integral

$$F_A^{(n)}(0) = \frac{n!}{2\pi} \int_{-\pi}^{\pi} F(e^{-j\theta} A) e^{jn\theta} d\theta , \qquad j = \sqrt{-1} . \tag{2.7}$$

Thanks to the w*-analyticity this integral defines the $\mathcal{B}$-valued linear w*-continuous form $\langle \Phi_n , A^{\otimes n} \rangle = F_A^{(n)}(0)/n!$ the positive definetness of which is a consequence of (i), if we approximate the Cauchy integral by an integral sum $\sum_{i=1}^{N} F(e^{-j\theta_i} A) e^{jn\theta_i} \Delta\theta_i$. Indeed, for any finite family of vectors $\zeta^m \in \mathcal{G}$ and $A_m \in \mathcal{A}$ the Hermitian form

$$\sum_{k,l=1}^{M} (\zeta^k | \langle \Phi_n , (A_k^* A_l)^{\otimes n} \rangle \zeta^l) = \frac{1}{2\pi} \sum_{k,l=1}^{M} (\zeta^k | \int_{-\pi}^{\pi} F(e^{-j\theta} A_k^* A_l) e^{jn\theta} d\theta \cdot \zeta^l) =$$

$$= \frac{1}{(2\pi)^2} \sum_{k,l=1}^{M} (\zeta^k | \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(e^{j(\alpha-\beta)} A_k^* A_l) e^{jn(\beta-\alpha)} d\alpha \, d\beta \cdot \zeta^l)$$

can be represented as the limit of positive integral sums

$$\frac{1}{(2\pi)^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sum_{k,l=1}^{M} (e^{jn\alpha} \zeta^k | F((e^{-j\alpha} A_k)^* (e^{-j\beta} A_l)) e^{jn\beta} \zeta^l) \, d\alpha \, d\beta =$$

$$= \frac{1}{(2\pi)^2} \lim_{N\to\infty} \sum_{p,q=1}^{M\cdot N} (\xi^p | F(X_p^* X_q) \xi^q) \geqslant 0 ,$$

where $\xi^{i+mN} = e^{jn\theta_i}\Delta\theta_i z^m$, $X_{i+mN} = e^{-j\theta_i}A_m$, $i = 1,\dots,N$,

m = 1, ... , M. This proves the positive-definetness of forms $A^{(n)} \in \mathcal{A}^{(n)} \mapsto \langle \Phi_n, A^{(n)}\rangle$ and, hence the complete positivity of the corresponding $\mathcal{B}$-valued w*-continuous form $\langle \Phi, \hat{A}\rangle = \sum \langle \Phi_n, A^{(n)}\rangle$ on the sum $\oplus \mathcal{A}^{(n)} = \mathcal{M}$ of W*-algebras $\mathcal{A}^{(n)}$, generated by $A^{\otimes n}$:

$$\sum_{k,l=1}^{M} (z^k | \langle \Phi_n, A_k^{(n)*} A_l^{(n)}\rangle z^l) = \sum_{p,q=1}^{N} (\xi^p | \langle \Phi_n, (A_p^* A_q)^{\otimes n}\rangle \xi^q) \geq 0$$

where $\xi^r = \sum_{m=1}^{M} c_m^r \zeta^m$, if $A_m^{(n)} = \sum_{r=1}^{N} c_m^r A_r^{\otimes n}$, $m = 1,\dots,M$ ▨

The proven theorem gives for an w*-analytical functional $f : \mathcal{A}^1 \to \mathbb{C}$ the necessary and sufficient condition (i) to be the generating functional for a positive form $\phi \in \mathcal{M}_*$, which together with the condition $f(I) \leq 1$ $(f(I)=1)$ characterizes the generating functionals of multiquantum normalized states over $\mathcal{A}$. In the case $F : \mathcal{A}^1 \to \mathcal{B}$ it gives together with $F(I) \leq I$ $(F(I)=I)$ the characterization of operator-valued (normalized) states $\mathcal{M} \to \mathcal{B}$ for the multiquantum system $\mathcal{M} = m(\mathcal{A})$, which are completely positive linear w*-continuous maps Φ , satisfying the contraction condition $\Phi(I^{\otimes}) \leq I$ $(\Phi(I^{\otimes}) = I)$. Using the Stinenspring representation $\pi_n : \mathcal{A}^{(n)} \to \mathcal{B}(\mathcal{K}_n)$ for each component as for a completely positive w*-continuous contractive map $\Phi_n(A) = J_n^* \pi_n(A) J_n$ we obtain a nonlinear extension $\mu(X) = \oplus \pi_n(X^{\otimes n})$, $J = \oplus J_n$ of the Kraus-Stinenspring construction for w*-analytical completely positive maps F. This gives, obviously, the following corollary of theorem 2.

Corollary I.2. Let $F_i : \mathcal{A}_i^1 \to \mathcal{B}_i$, $i = 1,\dots,N$ be a finite family of generating maps for $\mathcal{B}_i$-valued multiquantum states $\Phi_i : \mathcal{M}_i \to \mathcal{B}_i$ on the W*-algebras $\mathcal{M}_i = m(\mathcal{A}_i)$. Then the product $\otimes F_i$ as the map $\oplus X_i \mapsto \otimes F_i(X_i)$, defines the generating map $\oplus \mathcal{A}_i \to \otimes \mathcal{B}_i$ for an $\otimes \mathcal{B}_i$-valued multiquantum state over a decomposable W*-algebra $\oplus \mathcal{A}_i$. If $\mathcal{B}_{i+1} = \mathcal{A}_i$, $i < N$, then the composition $F_1 \circ \dots \circ F_N$ is defined on $\mathcal{A}^1 = \mathcal{A}_N^{(1)}$ as the map $X \mapsto F_1(\dots(F_N(X))\dots)$ into $\mathcal{B} = \mathcal{B}_1$, which is the generating map for a $\mathcal{B}$-valued multiquantum state over $\mathcal{A}$.

In particular, if $\{f^{(i)}\}$ is a finite family of generating functionals $f^{(i)}: \mathcal{A}^1 \to \mathbb{C}$ for a family of states $\varphi^{(i)}: \mathcal{M} \to \mathbb{C}$, then the product $f(X) = f^{(1)}(X) \cdots f^{(N)}(X)$ defines the generating functional $f = (\bigotimes_{i=1}^{N} f^{(i)}) \circ \iota$, $\iota(X) = X \otimes 1^{(N)}$ for the multiquantum state $\varphi: \mathcal{M} \to \mathbb{C}$ over $\mathcal{A}$, corresponding to the identification $\varphi = \omega \circ m(\iota)$ of the similar particles with types i = 1, ... , N in the multiquantum state $\omega: m(\mathcal{B}) \to \mathbb{C}$ over $\mathcal{B} = \mathcal{A} \otimes \mathbb{C}^N$ with generating functional $w(\bigoplus_{i=1}^{N} X_i) = f^{(1)}(X_1) \cdots f^{(N)}(X_N)$. In the tensor representation $\mathcal{N} = m(\mathcal{A})^{\otimes N}$ of multiquantum W*-algebra $m(\mathcal{A} \otimes \mathbb{C}^N)$ this state is the product state $\omega = \bigotimes_{i=1}^{N} \varphi^{(i)}$ because of its components ω_ν, $\nu = (n_1, \ldots, n_N)$ defined by Cauchy integrals

$$\langle \omega_\nu , \bigotimes_{i=1}^{N} A_i^{\otimes n_i} \rangle = \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} w(e^{-j\theta_1} A_1, \ldots, e^{-j\theta_N} A_N) e^{j \sum n_i \theta_i} d\theta_1 \cdots d\theta_N \quad (2.8)$$

have for $w(X_1, \ldots, X_N) = f^{(1)}(X_1) \cdots f^{(N)}(X_N)$ the tensor-product form $\omega_\nu = \bigotimes_{i=1}^{N} \varphi^{(i)}_{n_i}$, corresponding to

$$\langle \omega_\nu , \bigotimes_{i=1}^{N} A_i^{\otimes n_i} \rangle = \langle \varphi^{(1)}_{n_1} , A_1^{\otimes n_1} \rangle \cdots \langle \varphi^{(N)}_{n_N} , A_N^{\otimes n_N} \rangle$$

with

$$\langle \varphi_n , A^{\otimes n} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{-j\theta} A) e^{jn\theta} d\theta \quad . \qquad (2.9)$$

Hence the state φ is the convolution

$$\langle \varphi_n , A^{(n)} \rangle = \sum_{n_1 + \ldots + n_N = n} \langle \bigotimes_{i=1}^{N} \varphi^{(i)}_{n_i} , A^{(n)} \rangle = \langle (\circledast_{i=1}^{N} \varphi^{(i)})_n , A^{(n)} \rangle , \qquad (2.10)$$

as it follows from (1.3), and

$$\circledast_{i=1}^{N} \varphi^{(i)} = \bigoplus_{n < \infty} (\circledast_{i=1}^{N} \varphi^{(i)})_n = (\bigotimes_{i=1}^{N} \varphi^{(i)}) \circ \hat{\iota} , \qquad (2.11)$$

where $\hat{\imath} : \mathcal{M} \to \mathcal{M}^{\otimes N}$ is the multiplication w*-isomorphism $\bigoplus_{|\nu|<\infty} \imath^{(\nu)}$, defined by the embeddings

$$\imath^{(\nu)} : \mathcal{A}^{(|\nu|)} \to \mathcal{A}^{(n_1)} \otimes \ldots \otimes \mathcal{A}^{(n_N)} \qquad \text{for } \nu = (n_1, \ldots, n_N),\ |\nu| = n_1 + \ldots + n_N.$$

The simplest multiquantum state $\phi = [\varphi_n]$ with $\phi_n \neq 0$ for all n = 0,1, ... is the Poissonian state over $\mathcal{A}$ defined by the generating functional $f(X) = \langle \phi, X^{\otimes} \rangle$ of the invariant under multiplication form

$$f(X) = \exp\{ \langle \rho, X \rangle - c \} , \tag{2.12}$$

where ρ is a positive normal form $\rho \in \mathcal{A}_*^+$, and $c \geqslant \langle \rho, I \rangle$ ($c = \langle \rho, I \rangle$, if the state is normalized). The functional (2.12) generates the n-quantum states (2.9) in the form of product states

$$\phi_n = e^{-c} \rho^{\otimes n} / n! , \qquad n = 0, 1, \ldots \tag{2.13}$$

of a single-quantum state with Poissonian distribution

$$\langle \phi_n , I^{\otimes n} \rangle = e^{-c} \langle \rho , I \rangle^n / n! \qquad \text{if} \quad c = \langle \rho, I \rangle$$

of the quantum number n = 0,1, ... The composition of the multiquantum systems in Poissonian states ϕ^i over $\mathcal{A}_i$, $i = 1, \ldots, N$, described by $\ln f^{(i)}(X_i) = \langle \rho^{(i)}, X_i \rangle - c^{(i)}$ gives the Poissonian state over $\mathcal{B} = \bigoplus_{i=1}^N \mathcal{A}_i$ with $\ln w(\oplus X_i) = \sum (\langle \rho^{(i)}, X_i \rangle - c^{(i)})$, and the identification of the corresponding particles in the case $\mathcal{A}_i = \mathcal{A}$, $i = 1, \ldots, N$ induces the Poissonian state, described by (2.12) or (2.13) with

$$\rho = \sum_{i=1}^N \rho^{(i)} , \qquad c = \sum_{i=1}^N c^{(i)} . \tag{2.14}$$

3. The number and empirical weight operators and macroscopic observables

Let us consider a w*-dense $*$-subalgebra $\mathcal{O}\!\iota \subseteq \mathcal{A}$ of a one-particle von Neumann algebra $\mathcal{A} \leq \mathcal{B}(\mathcal{E})$ and identify the pre-dual space $\mathcal{A}_*$ with the subspace of linear forms $x \in \mathcal{O}\!\iota \mapsto \langle \rho, x \rangle$, defined by the restrictions $\rho | \mathcal{O}\!\iota$ of $\rho \in \mathcal{A}_*$. Due to the w*-density of $\mathcal{O}\!\iota$ in $\mathcal{A}$, the subspace $\mathcal{A}_*$ is separated by $\mathcal{O}\!\iota$ and is w*-dense in dual space $\mathcal{O}\!\iota^*$ of all linear bounded forms $\Psi : x \in \mathcal{O}\!\iota \mapsto \Psi(x)$, equipped with w*-topology, induced by $\mathcal{O}\!\iota$. Typically $\mathcal{O}\!\iota$ is a $*$-algebra, such that $\mathcal{A}_* = \mathcal{O}\!\iota^*$, as it is in the case of pure quantum particles, described by $\mathcal{A}_i = \mathcal{B}(\mathcal{E}_i)$ for quantum particles type $i \in J$, for which we define $\mathcal{O}\!\iota$ as the decomposable algebra of finite dimensional operators $x = \oplus x^{(i)}$ on the Hilbert space $\mathcal{E} = \oplus \mathcal{E}_i$. We define an <u>operator-valued weight</u> on $\mathcal{O}\!\iota$ describing the distribution of the number of quantums or particles over $\mathcal{A}$ as the linear functional $\hat{\nu} : x \in \mathcal{O}\!\iota \mapsto \hat{\nu}(x)$ with decomposable unbounded operator values $\hat{\nu}(x) = \bigoplus_{n=0}^{\infty} \hat{\nu}(x)^{(n)}$ on the Fock space $\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{E}^{\otimes n}$ with bounded components

$$\hat{\nu}(x)^{(n)} = \sum_{k=1}^{n} I^{\otimes(n-k)} \otimes x \otimes I^{\otimes(k-1)} = n\, I^{\otimes(n-1)} \circledS\, x \qquad (3.1)$$

on tensor powers $\mathcal{E}^{\otimes n}$ ($\circledS$ means the symmetrical tensor product). The map $\hat{\nu}$, called the <u>number weight operator over</u> $\mathcal{O}\!\iota$, gives an operator $*$-representation $\hat{\nu}(x^*) = \hat{\nu}(x)^*$ of the Lie algebra $[x, z] = xz - zx$, $x, z \in \mathcal{O}\!\iota$

$$\hat{\nu}([x,z]) = \hat{\nu}(x)\hat{\nu}(z) - \hat{\nu}(z)\hat{\nu}(x) \qquad (3.2)$$

of the group $\exp \mathcal{O}\!\iota \subset \mathcal{A}$. Of course it is normal and its extension on $\mathcal{A}$ maps the identity operator I into the total number operator $\hat{n} = \bigoplus_{n=0}^{\infty} n I^{\otimes n} = \hat{\nu}(I)$. Due to w*-density of corresponding group algebra $\exp\{\hat{\nu}(\mathcal{O}\!\iota)\}$ in $\mathcal{M} = \oplus \mathcal{A}^{(n)}$ on $\mathcal{F}$, the family $\hat{\nu}(\mathcal{O}\!\iota)$ is a generating family for the von Neumann algebra, representing $\mathcal{M}$ on $\mathcal{F}$.

In the case $\mathcal{A} = \oplus \mathcal{B}(\mathcal{E}_i)$ of pure quantum particles of different types one can obtain an equivalent (in the sense of w*-equivalence of generated von Neumann algebras) tensor representation on $\otimes \mathcal{F}_i$ for $\mathcal{F}_i = \bigoplus_{n=0}^{\infty} \mathcal{E}_i^{\otimes n}$:

$$\hat{\nu}\,(\underset{j\in J}{\oplus} x^{(j)}) \simeq \sum_{j\in J} \hat{\nu}_j\,(x^{(j)}) \otimes I_j^{\otimes} \;, \qquad x^{(j)} \in \mathcal{Ol}^{(j)}, \tag{3.3}$$

where $I_j^{\otimes} = \underset{|\nu_j|<\infty}{\oplus} I^{\otimes \nu_j}$, $\nu_j = (n_i)_{i\neq j}$, $I^{\otimes \nu_j} = \underset{i\neq j}{\otimes} I_i^{\otimes n_i}$, $\mathcal{Ol}^{(j)} \subseteq \mathcal{A}_j$ is the algebra of finite dimensional opertors on $\mathcal{E}_j$, and $\hat{\nu}_j\,(x^{(j)})^{(n)} = = n\, I_j^{\otimes(n-1)} \otimes x^{(j)}$, $n = n_j$ differs from zero for $\sum_{j\in J} n_j = |\nu| < \infty$ only on a finite subset of J.

Taking into account that

$$exp\,\{\hat{\nu}\,(x)\} = \overset{\infty}{\underset{n=0}{\oplus}} exp\,\{x\}^{\otimes n},$$

we find the symmetrical moment generating map (functional) for a $\mathcal{B}$-valued form (state, if $\mathcal{B}=\mathbb{C}$) $\Phi : \hat{A} \mapsto \langle \Phi, \hat{A}\rangle$ on $\mathcal{M}$

$$\Upsilon\,(x) = \langle \Phi,\, exp\{\hat{\nu}\,(x)\}\rangle = F\,(exp\,\{x\}\,) \tag{3.4}$$

as the composition of an exponential map $\mathcal{Ol} \to \mathcal{A}$ and the state generating map $F(X) = \langle \Phi, X^{\otimes}\rangle$, which is defined at least for all $x \in \mathcal{Ol}$ with $Re\, x = = (x + x^*)/2 \leq 0$, when $exp\,\{x\} \in \mathcal{A}^1$. But more useful and explicitly evaluated are the factorial moment generating maps (functionals)

$$\Psi\,(x) = \langle \Phi\,,\, (I + x)^{\otimes}\rangle = F\,(\,I + x\,)\,, \tag{3.4}$$

defined as expectations of the factorial exponential

$$:exp\{\hat{\nu}\,(x)\}: \; = \overset{\infty}{\underset{n=0}{\oplus}}\, (I + x)^{\otimes n}$$

at least for $x \in \mathcal{A}^1 - I$. The factorial moments $\Psi_x^{(m)} = \langle \Phi, :\hat{\nu}(x)^m:\rangle$,

$$:\hat{\nu}\,(x)^m: \; = \frac{d^m}{dt^m}\, :exp\{t\,\hat{\nu}\,(x)\}: \Big|_{t=0}\,, \tag{3.5}$$

$m=0,1,\dots,$ if they exist, are defined as linear $\mathcal{B}$-valued forms on the linear span of $\{x^{\otimes m}, x\in\mathfrak{A}\}$ which can be uniquely continued to the symmetrical forms on the algebraic tensor product $\mathfrak{A}^{\otimes m}$, spanning $x^{(1)}\otimes\dots\otimes x^{(m)}$ as

$$\langle \Psi^{(m)}, x^{(1)}\otimes\dots\otimes x^{(m)}\rangle = \frac{d^m}{dt_1\cdots dt_m}\Psi(t_1x^{(1)}+\dots+t_mx^{(m)})\Big|_{t=0}, \quad (3.6)$$

where $x^{(i)}\in\mathfrak{A}$. If these forms are bounded (w*-continuous), they are uniquely continued on C*-algebraic completions $\mathfrak{A}_m$ (W*-algebraic closure $\mathcal{A}^{(m)}$) of the span $\{x^{\otimes m}\}$ as the expectations $\langle\Phi, \hat{\nu}^{\odot m}(a_m)\rangle$ of factorial powers

$$\hat{\nu}^{\odot m}(a_m) = \bigoplus_{n=0}^{\infty}\hat{\nu}^{\odot m}(a_m)^{(n)}, \qquad a_m\in\mathfrak{A}_m,$$

$$\nu^{\odot m}(a_m)^{(n)} = n(n-1)\cdots(n-m+1)\, I^{\otimes(n-m)}\circledS a_m, \quad (3.7)$$

uniquely extending m-linear maps $x\mapsto :\hat{\nu}(x)^m:$ on $\mathfrak{A}$ to the normal linear maps on $\mathfrak{A}_m$ and $\mathcal{A}^{(m)}$ by $:\hat{\nu}(x)^m: = \hat{\nu}^{\odot m}(x^{\otimes m})$. One can easily find that

$$\hat{\nu}^{\odot 1}(x) = \hat{\nu}(x), \quad \hat{\nu}^{\odot 2}(x\otimes y) = \hat{\nu}(x)\hat{\nu}(y) - \hat{\nu}(xy), \quad (3.8)$$

and so on.

If $\lambda:\mathcal{A}\to\mathcal{B}$ is an identity preserving normal representation of $\mathcal{A}$ in a von Neumann algebra $\mathcal{B}\subseteq\mathcal{B}(\mathcal{G})$, then $m(\lambda)=\bigoplus_{n=0}^{\infty}\lambda^{\otimes n}$ defines a normal operator representation

$$m(\lambda)\circ\hat{\nu}(x) = \bigoplus_{n=0}^{\infty} n\, I^{\otimes(n-1)}\circledS\lambda(x) = \hat{\nu}\circ\lambda(x) \quad (3.9)$$

of Lie algebra $\mathfrak{A}$ on Fock space $\mathcal{H}=\bigoplus_{n=0}^{\infty}\mathcal{G}^{\otimes n}$, which can be considered as the composition of the number weight operator $\hat{\nu}$ over a w*-dense $*$-subalgebra $\mathcal{L}\subseteq\mathcal{B}$, $\lambda(\mathfrak{A})\subseteq\mathcal{L}$ on $\mathcal{H}=m(\mathcal{G})$ and λ. If $\mathcal{B}$ is a direct sum $\oplus\mathcal{A}_i$ of von Neumann algebras $\mathcal{A}_i\subseteq\mathcal{B}(\mathcal{E}_i)$, then there is an equivalent tensor representation on the Fock product $\otimes \mathcal{F}_i$ for the number weight operator $\hat{\nu}$ over $\mathcal{L} = \oplus \mathcal{O\!l}^{(i)}$ and all its factorial powers

$$\hat{\nu}^{\odot m}(b_m) \simeq \sum_{\mu : |\mu| = m} \frac{m!}{\mu!} \hat{\nu}^{\odot \mu}(b_\mu) , \tag{3.10}$$

where $b_m = \oplus_{j_1 \in J} \dots \oplus_{j_m \in J} a^{(j_1, \dots, j_m)}$ is an element of the C*-algebra $\mathcal{L}_m = \oplus \dots \oplus \mathcal{O\!l}^{(j_1 \dots j_m)}$ generated by $y^{\otimes m}$, $y = \oplus x^{(j)}$, $b_{\mu(j_1, \dots, j_m)} = u^{(m)\,-1}_{\mu(j_1, \dots, j_m)} (a^{(j_1, \dots, j_m)})$ is the unitary equivalent representation (1.5) of the component $\mathcal{O\!l}^{(j_1, \dots, j_m)} \subseteq \mathcal{B}(\mathcal{E}^{j_1} \otimes \dots \otimes \mathcal{E}^{j_m})$, generated by $x^{(j_1)} \otimes \dots \otimes x^{(j_m)}$ on $\mathcal{G}^{\otimes \mu} = \otimes_{i \in J} \mathcal{E}^{\otimes m_i}$, as C*-algebra $\mathcal{L}_\mu$, generated by $\otimes_{i \in J} a^{(i)}_{m_i}$, and

$$\hat{\nu}^{\odot \mu}(\otimes_{i \in J} a_{m_i}) = \otimes_{i \in J} \hat{\nu}_i^{\odot m_i}(a_{m_i}) \tag{3.11}$$

for $b_\mu = \otimes_{i \in J} a_{m_i}$, $a_{m_i} \in \mathcal{O\!l}_{m_i}$ with $\mu = (m_i)$, $|\mu| = \sum_{i \in J} m_i < \infty$.

In the case of the composition of N identical multiquantum systems over $\mathcal{A}_i = \mathcal{A}$, $i = 1, \dots, N$ the formula (3.10) gives for $b_m = a_m \otimes 1^{(mN)}$, $b_\mu = a_{|\mu|}$, where $1^{(mN)}$ is the identity operator on $\mathbb{C}^{mN}$ and $a_m \mapsto b_\mu$, $|\mu| = m$ is the imbedding $\mathcal{O\!l}_{|\mu|} \subseteq \mathcal{L}_\mu = \otimes \mathcal{O\!l}_{m_i}$, an equivalent representation on $\mathcal{F}^{\otimes N}$ of factorial powers for the number weight operator (2.9) over $\mathcal{O\!l} = \mathcal{O\!l}^{(i)}$, $i = 1, \dots, N$ with $\lambda(x) = x \otimes 1^{(N)}$, corresponding to the identification of similar particles with type index i=1, ... , N. In particular, for m=1 one can obtain

$$\hat{\nu}(x \otimes 1^{(N)}) \simeq \sum_{j=1}^{N} \hat{\nu}_j(x) \otimes I_j^{\otimes} = \hat{\imath}^{(N)} \circ \hat{\nu}(x) \tag{3.12}$$

as in the case $x^{(j)} = x$, $j = 1, \dots, N$ for (3.3). This gives the equivalent tensor representation for the normed weight operator

$$\hat{\rho}(x) = \hat{\nu}(x \otimes 1^{(N)})/N = m(1^{(N)}) \circ \hat{\nu}(x/N) , \tag{3.13}$$

$\iota^{(N)}(x) = x \otimes 1^{(N)}$, and its factorial powers

$$\hat{\rho}^{\odot m}(x^{\otimes m}) = :(\hat{\nu}(x \otimes 1^{(N)})/N)^m : = \hat{\nu}^{\odot m}(x^{\otimes m} \otimes 1^{(mN)})/N^m$$

as the empirical weight operator

$$\hat{\rho}_{emp}(x) = \frac{1}{N}\sum_{j=1}^{N} \hat{\nu}_j(x) \otimes I_j^{\otimes} \simeq \hat{\rho}(x), \quad x \in \mathcal{A} \tag{3.14}$$

and its factorial powers $\hat{\iota}^{(N)} \circ \hat{\nu}^{\odot m}(a_m/N^m)$,

$$\hat{\rho}^{\odot m}_{emp}(a_m) = \frac{1}{N^m}\sum_{\mu:|\mu|=m} \frac{m!}{\mu!} \hat{\nu}^{\odot\mu}(a_{|\mu|}), \quad a_m \in \mathcal{A}_m \tag{3.15}$$

in the composed system on the Hilbert space $\mathcal{F}^{\otimes N}$.

Now we can consider an algebra of macroscopic observables as the empirical representation in N-component multiquantum system with identical $\mathcal{A}_i = \mathcal{A}$ of a $*$-algebra, generated by the normed weight operator (2.13). We define a macroscopic observable over $\mathcal{A}$ as a factorial power series of the normed weight operator (3.13):

$$\hat{Y} = \sum_{m=0}^{\infty} \hat{\rho}^{\odot m}(a_m) = \sum_{m=0}^{\infty} \langle \hat{\rho}^{\odot m}, a_m \rangle, \tag{3.16}$$

where we use the bracket notation $\langle \hat{\rho}^{\odot}, a \rangle = \hat{\rho}^{\odot}(a)$ for an operator-valued functional $\hat{\rho}^{\odot}$ to emphasize its linearity. As it follows from the definitions (3.7), (3.13), the observable (3.16) is the representation $m(\iota^{(N)})$ on $\mathcal{H} = \oplus \mathcal{G}^{\otimes n}$, $\mathcal{G} = \mathcal{E} \otimes \mathbb{C}^N$ of a decomposable operator $\hat{X} = \bigoplus_{n=0}^{\infty} X^{(n)}$ with finite sum components $X^{(n)} = \sum_{m=0}^{\infty} \hat{\nu}^{\odot m}(a_m/N^m)^{(n)}$:

$$X^{(n)} = \sum_{m=0}^{n} n(n-1)\cdots(n-m+1)\, I^{\otimes(n-m)} \circledS a_m/N^m. \tag{3.17}$$

Hence in order to study the algebra of macroscopic observables we have to study the algebra of decomposable operators on Fock space $\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{E}^{\otimes n}$ depending on $\varepsilon = 1/N$ as

$$\hat{X} = \sum_{m=0}^{\infty} \varepsilon^m \langle \hat{\nu}^{\otimes m}, a_m \rangle \equiv :\alpha(\varepsilon\hat{\nu}): \tag{3.18}$$

with bounded components (3.17), defined by some sequences $a = [a_m]$, $a_m \in \mathcal{O}\!\mathcal{l}_m$. The functional $\alpha : \mathcal{A}_* \to \mathbb{C}$, defined on $\mathcal{A}_* \subseteq \mathcal{O}\!\mathcal{l}^*$ by formal series

$$\alpha(\rho) = \sum_{m=0}^{\infty} \langle \rho^{\otimes m}, a_m \rangle , \qquad \rho \in \mathcal{A}_* \tag{3.19}$$

is called the symbol of the observable (3.18). Let us introduce an inductive topology on the linear space of all sequences $a = [a_m]$ with $\|a_m\| \leqslant \leqslant c\, \xi^m / m!$, $\xi \in \mathbb{R}^+ \ni c = c(a)$. We denote by $\mathfrak{M}^{(\xi)}$ the Banach space of the sequences with norm

$$\|a\|^{(\xi)} = \sup_m m!\, \|a_m\| / \xi^m , \tag{3.20}$$

and $\mathfrak{M}^{(\infty)} = \bigcup_{\xi > 0} \mathfrak{M}^{(\xi)}$ the inductive limit, in which $a^{(\varepsilon)} \to 0$ for $\varepsilon \to 0$, if $\|a^{(\varepsilon)}\|^{\xi} \to 0$ for a $\xi > 0$. Now we shall prove that the operator representation (3.19) of $\mathfrak{M}^{\xi}$ on $\mathcal{F} = \oplus\, \mathcal{E}^{\otimes n}$ induces in $\mathfrak{M}^{(\xi)}$ the structure of an inductive $*$-algebra, with the product $\circledast$ analytically depending on $\varepsilon \in \mathbb{R}^+$, which is commutative in the limit $\varepsilon \to 0$. In such a way we shall prove, that the $*$-algebra of macroscopic observables (3.16) is asymptotically Abelian, if $N \to \infty$.

Theorem I.3. Let $\hat{X} = :\alpha(\varepsilon\hat{\nu}):$ and $\hat{Z} = :\gamma(\varepsilon\hat{\nu}):$ be the operators of the form (3.18), defined by $a = [a_m] \in \mathfrak{M}^{(\xi)}$ and $c = [c_m] \in \mathfrak{M}^{(\zeta)}$ respectively. Then the product $\hat{Y} = \bigoplus_{n=0}^{\infty} X^{(n)} Z^{(n)}$ is an operator $\hat{Y} = :\beta(\varepsilon\hat{\nu}):$ of the same form (3.18), defined by $b^{(\varepsilon)} = [b_m^{(\varepsilon)}]$ with

$$b_m^{(\varepsilon)} = \sum_{i+j+k=m} (i+j)!\, \varepsilon^j (j+k)!\, a_{i+j} \underset{j}{\overset{\odot}{\circ}} c_{j+k} / i!j!k! \equiv \tag{3.21}$$

$$\equiv (a \circledcirc_{\varepsilon} c)_m, \qquad m = 0, 1, 2 \ldots,$$

where $a_{i+j} \underset{j}{\overset{\odot}{\circ}} c_{j+k}$ is a semi tensor product, defined for

$a_{i+j} = a_i \circledS a_j \qquad c_{k+j} = c_k \circledS c_j$

by the symmetrical tensor product $\circledS$ as

$$a_i \circledS a_j \underset{j}{\overset{\odot}{\circ}} c_k \circledS c_j = a_i \circledS a_j c_j \circledS c_k, \tag{3.22}$$

and $\| b^{(\varepsilon)} \|^{(\eta)} \leq \| a \|^{(\xi)} \| c \|^{(\zeta)}$, if $\eta \geq \xi + \eta + \varepsilon \xi \zeta$.

If $\varepsilon \to 0$ then $b^{(\varepsilon)} \to b^{(0)}$, where $b^{(0)} = a \circledast c$ is the convolution with components

$$(a \circledast c)_m = \sum_{i+k=m} a_i \circledS c_k = \lim_{\varepsilon \to 0} (a \circledcirc_{\varepsilon} c)_m, \tag{3.23}$$

and if $\varepsilon < 1$, $\eta \geq \xi + \zeta + \xi \zeta$ then

$$\| b^{(\varepsilon)} - b^{(0)} \|^{(\eta)} \leq \varepsilon \| a \|^{(\xi)} \| c \|^{(\zeta)} / (1 - \varepsilon). \tag{3.24}$$

Proof. Let $a = [x^{\otimes m}/m!]$, $c = [z^{\otimes m}/m!]$ with $\|x\| \leq \xi$, $\|z\| \leq \zeta$. We have, obviously, $\|a\|^{(\xi)} \leqslant 1$, $\|c\|^{(\zeta)} \leqslant 1$ and $\mathfrak{M}^{(\xi)}$, $\mathfrak{M}^{(\zeta)}$ are the completions of linear spans of such $a \in \mathfrak{M}^{(\xi)}$ and $c \in \mathfrak{M}^{(\zeta)}$. Hence we have to prove (3.21) only for $a_m = x^{\otimes m}/m!$ and $c_m = z^{\otimes m}/m!$. But in this case

$$X^{(n)} = (I + \varepsilon x)^{\otimes n}, \quad Z = (I + \varepsilon z)^{\otimes n},$$

following from the definition (3.12), and

$$Y^{(n)} = X^{(n)} Z^{(n)} = [(I + \varepsilon x)(I + \varepsilon z)]^{\otimes n} = (I + \varepsilon(x+z) + \varepsilon^2 xz)^{\otimes n}.$$

So the representation of $\hat{Y} = \oplus Y^{(n)}$ in the form (3.18) is defined by coefficients

$$b_m^{(\varepsilon)} = (x + z + \varepsilon x z)^{\otimes m}/m! = \sum \frac{\varepsilon^j}{i!j!k!} x^{\otimes i} \circledS (xz)^{\otimes j} \circledS z^{\otimes k}, \tag{3.24}$$

which are the product (3.21) for $a_{i+j} = x^{\otimes (i+j)}/(i+j)!$, and $c_{j+k} = z^{\otimes (j+k)}/(j+k)!$, with

$$\|b_m^{(\varepsilon)}\| \leqslant (\|x\| + \|z\| + \varepsilon \|x\| \cdot \|z\|)^m/m! \leqslant (\xi + \zeta + \varepsilon \xi \zeta)^m/m!.$$

This proves the first part of the theorem with

$$\|a \odot c\|^{(\eta)} \leqslant \|a\|^{(\xi)} \|c\|^{(\zeta)} \qquad \text{for} \quad \eta \geqslant \xi + \zeta + \varepsilon \xi \zeta.$$

Now let us estimate the difference between (3.21) and (3.23) for $\varepsilon < 1$. Representing it as

$$b_m^{(\varepsilon)} - b_m^{(0)} = \varepsilon \sum_{j=1}^{m} \sum_{i+k=m-j} \frac{(i+j)!\,\varepsilon^{j-1}(j+k)!}{i!\,j!\,k!} a_{i+j} \odot c_{j+k},$$

we obtain for $a \in \mathcal{M}^{(\xi)}$, $c \in \mathcal{M}^{(\zeta)}$:

$$\| b_m^{(\varepsilon)} - b_m^{(0)} \| \leq \varepsilon \|a\|^{(\xi)} \|c\|^{(\zeta)} \sum_{j=1}^{m} \sum_{i+k=m-j} \varepsilon^{j-1} \xi^{i+j} \zeta^{j+k} / i!\,j!\,k! =$$

$$= \varepsilon \|a\|^{(\xi)} \|c\|^{(\zeta)} [(\xi+\zeta+\varepsilon\xi\zeta)^m - (\xi+\zeta)^m] / \varepsilon \; .$$

Using the interpolation Newton formula, we have

$$\| b_m^{(\varepsilon)} - b_m^{(0)} \| \leq \varepsilon \|a\|^{(\xi)} \|c\|^{(\zeta)} m \xi\zeta (\xi+\zeta+\theta\xi\zeta)^{m-1} ,$$

where $0 < \theta < \varepsilon$. Taking into account that

$$m \xi\zeta (\xi+\zeta+\theta\xi\zeta)^{m-1} \leq (\xi+\zeta+\xi\zeta)^m / (1-\theta)$$

for $0 \leq \theta < 1$ and $(1-\theta)^{-1} \leq (1-\varepsilon)^{-1}$ if $\varepsilon < 1$, we obtain the estimation (3.24):

$$\| b_m^{(\varepsilon)} - b_m^{(0)} \| \leq \varepsilon \|a\|^{(\xi)} \|b\|^{(\zeta)} (\xi+\zeta+\xi\zeta)^m / (1-\varepsilon)$$

<u>Corollary</u> I.3. <u>The inductive $*$-algebra</u> $\mathcal{M}^{(\infty)}$ <u>of macroscopic observables is isomorphic to the unital $*$-algebra of holomophic functionals on</u> $\mathcal{A}_*$ <u>with derivatives</u>

$$\delta_{\sigma}^{(m)} \alpha(\rho) = \frac{m!}{2\pi} \int_{-\pi}^{\pi} \alpha(\rho + e^{-j\theta}\sigma) e^{jm\theta} d\theta = \langle \sigma^{\otimes m}, \alpha^{(m)}(\rho) \rangle , \qquad (3.25)$$

$\sigma \in \mathcal{A}_*$, <u>uniquely extendable to the w*-continuous functionals</u> $\alpha^{(m)}(\rho) : \mathcal{O}\!\mathcal{L}_m \to \mathbb{C}$ <u>for every</u> ρ <u>, having the norms</u> $\|\alpha^{(m)}(0)\| \leq c\,\xi^m$ <u>at</u> $\rho = 0$ <u>for a</u> $\xi \in \mathbb{R}^+$. <u>The unity, involution, and product on this functional algebra is defined by</u>

$\alpha(\rho)=1$, $\alpha^*(\rho)=\alpha(\rho^*)^*$, and

$$(\alpha \circledcirc \gamma)(\rho)=\sum_{m=0}^{\infty} \frac{\varepsilon^m}{m!} \langle \rho^{\otimes m}, \alpha^{(m)}(\rho)\cdot\gamma^{(m)}(\rho)\rangle, \tag{3.26}$$

where the product $\alpha^{(m)}\cdot\gamma^{(m)}$ is defined $\alpha^{(m)}(\rho)$ and $\gamma^{(m)}(\rho)$ as for elements of C*-algebras $\mathfrak{A}_m$. In particular, for $\varepsilon=0$ $(\alpha\circledcirc\gamma)(\rho)=\alpha(\rho)\gamma(\rho)$.

Indeed, if $a \in \mathfrak{M}^{(\infty)}$, then the series (3.20) also lately converges on $\mathcal{A}_*$, and

$$|\alpha(\rho)| \leq \sum_{m=0}^{\infty} \varepsilon^m \|\rho\|^m \|a_m\| \leq \|a\|^{(\xi)} \exp\{\varepsilon\xi\|\rho\|\} .$$

Hence the derivatives (3.25) are defined for all ρ , as the absolutely convergent series

$$\langle \sigma^{\otimes m}, \alpha^{(m)}(\rho)\rangle = \sum_{k=0}^{\infty} \langle \sigma^{\otimes m} \circledS \rho^{\otimes k}, a_{m+k}\rangle (m+k)!/k! =$$

$$= \langle \sigma^{\otimes m}, \sum_{k=0}^{\infty} \langle \rho^{\otimes k}, a_{m+k}\rangle (m+k)!/k! \rangle .$$

To prove formula (3.25), derived in [4] , it is sufficient to check it for exponential functionals $\alpha(\rho)=\exp\{\langle\rho,x\rangle\}$, which are symbols of generating elements $a=[x^{\otimes m}/m!]$. Indeed, the symbol of the product (3.24) of such elements with $c=[z^{\otimes m}/m!]$ is

$$\beta^{\varepsilon}(\rho) = \sum_{m=0}^{\infty} \langle \rho^{\otimes m}, b_m^{(\varepsilon)}\rangle = \exp\{\langle\rho, x+z+xz\varepsilon\rangle\} =$$

$$= \sum_{m=0}^{\infty} \frac{\varepsilon^m}{m!} \langle \rho^{\otimes m}, (xz)^{\otimes m}\rangle \exp\{\langle\rho, x+z\rangle\} =$$

$$= \sum_{m=0}^{\infty} \frac{\varepsilon^m}{m!} \langle \rho^{\otimes m}, (\delta^{\otimes m} e^{\langle\rho,x\rangle})(\delta^{\otimes m} e^{\langle\rho,z\rangle})\rangle .$$

Chapter II.

Quantum point stochastic processes and kinetic equations

1. Quasi-Poissonian quantum processes and infinitely divisible multiquantum states

Let us consider one-parameter family $\{\mathcal{B}_t\}$ of von Neumann algebras $\mathcal{B}_t \subseteq \mathcal{B}(\mathcal{G}_t)$ on an increasing family of Hilbert spaces $\mathcal{G}_t$, $t \in \mathbb{R}^+$, satisfying the decomposability conditions

$$\mathcal{B}_{r+s} = \mathcal{B}_r \oplus \mathcal{B}_s \quad , \quad \mathcal{G}_{r+s} = \mathcal{G}_r \oplus \mathcal{G}_s \quad , \forall r,s \in \mathbb{R}^+, \tag{1.1}$$

where $\mathcal{B}_0 = \{0\} = \mathcal{G}_0$. Defining the multiquantum W*-algebras $\mathcal{N}_t = m(\mathcal{B}_t)$ on Fock spaces $\mathcal{H}_t = \oplus\, \mathcal{G}_t^{\otimes n}$, we can consider $\{\mathcal{N}_t\}$ as an increasing family of von Neumann algebras with $\mathcal{N}_0 = \mathbb{C} = \mathcal{H}_0$ defining the embeddings of $\mathcal{N}_t$ into $\mathcal{N}_{t+s}$ by canonical maps $\hat{B} \in \mathcal{N}_t \mapsto \hat{B} \otimes I_s^{\otimes}$ in the W*-representation $\mathcal{N}_t \otimes \mathcal{N}_s$ of $\mathcal{N}_{t+s}$ on $\mathcal{H}_t \otimes \mathcal{H}_s$.

A family of states $\{\omega^t\}$ on the increasing family $\{\mathcal{N}_t\}$ is called quasi-Poissonian, if it satisfies the product condition

$$\omega^{r+s} = \omega^r \otimes \omega^s \quad , \qquad \forall r,s \in \mathbb{R}^+ \tag{1.2}$$

under the W*-tensor representation $\mathcal{N}_r \otimes \mathcal{N}_s$ of $\mathcal{N}_{r+s}$. An equivalent definition of a quantum quasi-Poissonian process in terms of state generating functionals on $\mathcal{B}_t^1 = \{Y \in \mathcal{B}_t \mid Y^*Y \leq I\}$

$$w^t(Y) = \sum_{n=0}^{\infty} \langle \omega_n^t , Y^{\otimes n} \rangle \equiv \langle \omega^t, Y^{\otimes} \rangle ,$$

is the multiplicity condition

$$w^{r+s}(X \oplus Z) = w^r(X)\, w^s(Z) \qquad , \forall r,s \in \mathbb{R}^+ \tag{1.3}$$

for all $X \in \mathcal{B}_r^1$, $Z \in \mathcal{B}_s^1$.

Let us consider a continuous quasi-Poissonian process, defined for the Hilbert integrals $\mathcal{G}_t = \int_{s<t}^{\oplus} \mathcal{E}(s)\,ds$ of copies $\mathcal{E}(t) = \mathcal{E}$ of a Hilbert space $\mathcal{E}$

and the decomposable W*-algebras $\mathcal{B}_t = \int_{s\leq t}^{\oplus} \mathcal{A}(s)ds$ with $\mathcal{A}(s) = \mathcal{A}$ by generating functions

$$w^t\left(\int_{s\leq t}^{\oplus} X(s)ds\right) = \exp\left\{\int_0^t \ell(X(s))ds\right\}, \tag{1.4}$$

where $X : (0,t] \to \mathcal{A}^1$ is a measurable essentially bounded function as an element of the von Neumann algebra $\mathcal{B}_t = \mathcal{A} \otimes \mathcal{L}^\infty(0,t]$ on $\mathcal{G} = \mathcal{E} \otimes \mathcal{L}^2(0,t]$. Obviously, such an exponential family $[w^t]$ is a continuous family, satisfying the condition (1.3) with $X = \int_{t\leq r}^{\oplus} X(t)dt$, $Z = \int_{t\leq s}^{\oplus} X(r+t)dt$ and the w*-analytical functional $\ell : \mathcal{A}^1 \to \mathbb{C}$ is defined as the generator $\ell(X) = dp(t,X)/dt \mid t=0$ of a one-parameter multiplicative semigroup $\{p(t)\}$ of generating functionals

$$p(t,X) = w^t(X \otimes \mathcal{X}(t)) = \exp\{t\,\ell(X)\},\ t \in \mathbb{R}^+, \tag{1.5}$$

where $\mathcal{X}(t,s) = 1$ if $s \leq t$, and $\mathcal{X}(t,s) = 0$ if $s > t$, is the identity of the commutative W*-algebra $\mathcal{L}^\infty(0,t]$. An example of such a continuous process is the family $\{w^t\}$ of generating functionals (1.4) for Poissonian multiquantum states over $\mathcal{B} = \int_{s\leq t}^{\oplus} \mathcal{A}(s)ds$ defined by $\ell(X) = \langle \rho, X\rangle - c$:
$\ln w^t\left(\int_{s\leq t}^{\oplus} X(s)ds\right) = \int_0^t \langle \rho, X(s)\rangle ds - ct$ with $\rho \in \mathcal{A}_*^+$ and $c \geq \langle \rho, I\rangle$.

The necessary equivalent conditions for an w*-analytical functional $\ell(X)$ to be the generating functional of a one-parameter multiplicative semigroup of state generating functions are given by the following theorem for the scalar case $\mathcal{B} = \mathbb{C}$, $\mu(X) = 1$. It is a nonlinear generalization of the Evans-Lewis theorem [2] for linear conditionally positive maps $L : \mathcal{A} \to \mathcal{B}$ with respect to $\mu(X) = X$ for $\mathcal{B} = \mathcal{A}$ and $L^*(X) = L(X^*)^*$.

Theorem II.1. Let the derivative

$$L(X) = dP(t,X)/dt \mid t=0 \tag{1.6}$$

for a family $\{P(t) \mid t \in \mathbb{R}^+\}$ of positive-definite maps $P(t) : \mathcal{A}^1 \to \mathcal{B}$ be defined in the W*-algebra unit ball $\mathcal{A}^1$ as an w*-analytical map into a von Neumann algebra $\mathcal{B} \subseteq \mathcal{B}(\mathcal{G})$ and $P(0) = \mu$ be an w*-analytical representation (I. 2.4) of the multiplicative *-semigroup $\mathcal{A}^1$ in $\mathcal{B}$ with $\mu(I) = I$.

Then it satisfies the following equivalent conditions:

(i) $L=L^*$ is conditionally positive-definite (C.P.D.)

$$\sum_{k,l=1}^{M} (\zeta^k | L(X_k^* X_l)\zeta^l) \geqslant 0 \; , \qquad \forall \zeta^m \in \mathcal{G}, X_m \in \mathcal{A}^1, \; m \leqslant M$$

for all $M = 1,2, \ldots$, if $\sum_{m=1}^{M} \mu(X_m)\zeta^m = 0$.

(ii) $L=L^*$ is conditionally completely positive, i.e. $\hat{L} : \hat{X} \mapsto [L(X_l^k)]$ is conditionally positive:

$$(\hat{\zeta} \,|\, \hat{L}(\hat{X}^* \hat{X})\hat{\zeta}) \geqslant 0 \; , \qquad \forall \hat{\zeta} \in \hat{\mathcal{G}}, \hat{X} \in \hat{\mathcal{A}}^1,$$

if $\hat{\mu}(\hat{X})\hat{\zeta} = 0$, where $\hat{\zeta} \in \mathcal{G} \otimes \ell^2$ and $\hat{\mu}(\hat{X}) = [\mu(X_l^k)]$ for $\hat{X} = [X_l^k]$

(iii) $$L(X) = F(X) - B^* \mu(X) - \mu(X) B \; ,$$

where F is a w*-analytical completely positive map $\mathcal{A}^1 \to \mathcal{B}$ and $B \in \mathcal{B}$.

(iv) $$L(X) = \sum_{n=0}^{\infty} \langle \Lambda_n , X^{\otimes n} \rangle \equiv \langle \Lambda , X^{\otimes} \rangle ,$$

where $\Lambda = [\Lambda_n]$ is a linear w*-continuous conditionally completely positive $\mathcal{B}$-valued *-form on $\mathcal{M} = m(\mathcal{A}) = \oplus \mathcal{A}^{(n)}$, $\langle \Lambda , \hat{A}^* \rangle = \langle \Lambda , \hat{A} \rangle^*$, $\forall \hat{A} \in \mathcal{M}$.

Proof: If $P(t)$ are positive-definite maps $\mathcal{A}^1 \to \mathcal{B}$ in the sense of (i) in Theorem I.1, then the differences $P(t) - \mu$ and hence the derivative (1.6) as the limit $L(X) = \lim_{t \downarrow 0} P(t,X) - \mu$ are conditionally positive definite:

$$\sum_{m=1}^{M} \mu(X_m)\zeta^m = 0 \Rightarrow \sum_{k,l=1}^{M} (\zeta^k | (P(t, X_k^* X_l) - \mu(X_k^* X_l))\zeta^l) =$$

$$= \sum_{k,l=1}^{M} (\zeta^k | P(t, X_k^* X_l)\zeta^l) - \| \sum_{m=1}^{M} \mu(X_m)\zeta^m \|^2 \geqslant 0 \; .$$

As the implicatins (iii) $\Rightarrow$(ii) $\Rightarrow$(i) are evidently verifiable, and (iv) $\Rightarrow$(iii) follows directly from the Lewis-Evans theorem for cocycles on a von Neumann algebra $\mathcal{M} = \oplus \mathcal{A}^{(n)}$ with respect to a normal representation $\pi : \mathcal{M} \to \mathcal{B}$, defining $\mu(X) = \pi(X^{\otimes})$, we have to prove only the implication (i) $\Rightarrow$(iv). Due to the w*-analyticity of the map $L : \mathcal{A}^1 \to \mathcal{B}$ there exists a unique linear w*-continuous map $\Lambda : m(\mathcal{A}) \to \mathcal{B}$, for which $L(X) = \langle \Lambda, X^{\otimes} \rangle$. In the same way we can obtain a unique w*-continuous map $\pi : m(\mathcal{A}) \to \mathcal{B}$, for which $\mu(X) = \pi(X^{\otimes})$, extending the n-linear components

$$\mu_n(A) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mu(e^{-j\theta} A) e^{jn\theta} d\theta, \qquad j = \sqrt{-1} \tag{1.7}$$

of the analytical map $\mu(X) = \sum_{n=0}^{\infty} \mu_n(X)$ to linear components π_n:

$$\mathcal{A}^{(n)} \to \mathcal{B}, \qquad \pi_n\Big(\sum_{i=1}^{N} c^i A_i^{\otimes n}\Big) = \sum_{i=1}^{N} c^i \mu_n(A_i)$$

of the w*-continuous map $\pi(\hat{A}) = \sum_{n=0}^{\infty} \pi_n(A^{(n)})$. But the linear map π proves to be a W*-representation of $m(\mathcal{A}) = \oplus \mathcal{A}^{(n)}$ into $\mathcal{B}$ due to the orthogonality of μ_m and μ_n, and hence π_m and π_n under $m \neq n$ and its multiplicity under $n = m$: for all $A, B \in \mathcal{A}$

$$\mu_m(A)^* \mu_n(B) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \mu(e^{j\alpha} A^*) e^{j(n\beta - m\alpha)} \mu(e^{-j\beta} B)\, d\alpha\, d\beta =$$

$$= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \mu(e^{j(\alpha-\beta)} A^* B) e^{j(n\beta - m\alpha)} d\alpha\, d\beta =$$

$$= \delta_m^n \frac{1}{2\pi} \int_{-\pi}^{\pi} \mu(e^{-j\theta} A^* B) e^{jn\theta} d\theta = \delta_m^n \mu_n(A^* B).$$

Now we shall prove the property of C.P.D. for Λ with respect to the representation π of the linear span $\{\sum c^r X_r^{\otimes}\}$ and, hence, conditionally complete positivity of Λ with respect to π, due to the w*-density of this span in $m(\mathcal{A})$. Indeed, if $\sum_{m=1}^{M} \pi(\hat{A}_m) \zeta^m = 0$ for a family $\{\zeta^m \in \mathcal{G}\}$

and $\{\hat{A}_m \in m(\mathcal{A})\}$ of the form $\hat{A}_m = \sum_{r=1}^{N} c_m^r X_r^{\otimes}$, $m = 1,\dots,M$, then

$$\sum_{k,l=1}^{M} (z^k | \langle \Lambda, \hat{A}_k^* \hat{A}_l \rangle z^l) = \sum_{p,q=1}^{N} (\xi^p | \langle \Lambda, (X_p^* X_q)^{\otimes} \rangle \xi^q) \geq 0,$$

where $\xi^r = \sum_{m=1}^{M} c_m^r \zeta^m$ for a C.P.D. $L(X) = \langle \Lambda, X^{\otimes} \rangle$ with respect to $\mu(X) = \pi(X^{\otimes})$, since

$$\sum_{r=1}^{N} \mu(X_r)\xi^r = \sum_{r=1}^{N} \pi(X_r^{\otimes})\xi^r = \sum_{m=1}^{M} \pi(\hat{A}_m) z^m = 0 .$$

But it is well-known (see, for example [2]) that the linear w*-continuous map $\Lambda = \Lambda^*$ on a W*-algebra $\mathcal{M} = m(\mathcal{A})$ into another W*-algebra $\mathcal{B}$ is conditionally positive definite with respect to a W*-representation $\pi : \mathcal{M} \to \mathcal{B}$ iff it has the form

$$\langle \Lambda, \hat{A} \rangle = \langle \Phi, \hat{A} \rangle - \pi(\hat{A})B - B^*\pi(\hat{A}) , \qquad (1.8)$$

where Φ is a w*-continuous completely positive map $\mathcal{M} \to \mathcal{B}$, and $B \in \mathcal{B}$ is a bounded operator on $\mathcal{G}$. In such a way we obtain the structure (iii) for $L(X) = \langle \Lambda, X^{\otimes} \rangle$, where $F(X) = \langle \Phi, X^{\otimes} \rangle$, and $\mu(X) = \pi(X^{\otimes})$ ▨

<u>Corollary</u> II.1. <u>Let us consider the scalar case</u> $\mathcal{B} = \mathbb{C} = \mathcal{G}$ <u>with w*-analytical</u> $f : \mathcal{A}^1 \to \mathbb{C}$. <u>Then the necessary equivalent conditions</u>

(i) $\sum_{m=1}^{M} c^m = 0 \Rightarrow \sum_{k,l=1}^{M} \bar{c}^k \ell(X_k^* X_l) c^l \geq 0$, $\forall c^m \in \mathbb{C}$, $X_m \in \mathcal{A}^1$, $M = 1, 2, \dots$

(ii) $\hat{1} \cdot \hat{c} = 0 \Rightarrow \hat{c}^* \ell(\hat{X}^* \hat{X}) \hat{c} \geq 0$, $\forall \hat{c} \in \ell^2$, $\hat{X} \in \hat{\mathcal{A}}^1$, $\hat{1} = [1, 1, \dots]$

(iii) $\ell(X) = f(X) - c$, <u>where</u> $f : \mathcal{A}^1 \to \mathbb{C}$ <u>is a</u> P.D. <u>w*-analytical functional, and</u> $c \in \mathbb{C}$, <u>for</u> ℓ <u>to be the logarithm</u> $\ell(X) = \ln p(X)$ <u>an infinitely divisible state generating functional</u> $p(X)$, <u>i.e. to be the</u>

generator for a multiplicative semigroup $p(t,X)=p(X)^t$, $t>0$, are also sufficient together with $\ell(I)\leq 0 \Leftrightarrow f(I)\leq c \Rightarrow f^*=f$ (or $\ell(I)=0$ if $p(t)$ is normalized: $p(t,I)=1$ for a $t\in\mathbb{R}^+$).

Indeed, due to the condition (iii)

$$p(t,X)=\exp\{t\,\ell(X)\}=e^{-ct}\sum_{k=0}^{\infty}\frac{t^k}{k!}f(X)^k \tag{1.9}$$

is for any $t\in\mathbb{R}^+$ an w*-analytical P.D. - functional as the sum of P.D. functionals $t^k f(X)^k/k!$, $k=0,1,\dots$, and, obviously $p(t,I)\leq 1$ $(p(t,I)=1)$ is equivalent to $\ell(I)\leq 0$ $(\ell(I)=0)$ or $f(I)\leq c$ $(f(I)=c)$. So the generating functionals (1.4) of a continuous quasi-Poisson family of states $\{\omega^t\}$ have the structure $\exp\{\int_0^t f(X(s))ds - ct\}$:

$$w^t\Big(\int_{s\leq t}^{\oplus} X(s)ds\Big)=e^{-ct}\sum_{k=0}^{\infty}\int_{t_{k-1}}^{t}\dots\int_{t_0}^{t} f(X(t_k))dt_k\cdots f(X(t_1))dt_1, \tag{1.10}$$

corresponding to the decomposition $\omega^t=e^{-ct}\cdot\bigoplus_{k=0}^{\infty}\varphi^{\otimes k}\otimes\lambda_k^t$, $\langle\varphi,X^{\otimes}\rangle=f(X)$,

$$\langle\lambda_k^t,\varphi\rangle=\int_{t_{k-1}}^{t}\dots\int_{t_0}^{t}\varphi(t_1,\dots,t_k)\,dt_1\cdots dt_k,\quad\forall\varphi\in\mathcal{L}^{\infty}(\Omega_k^t)$$

of the state ω^t in the direct sum of tensor powers of a $\phi\in\mathcal{M}_*^+$ multiplied on the natural measures λ_k^t on the simplex-spaces $\Omega_k^t=\{(t_1,\dots,t_k)|0<t_1\dots<t_k\leq t\}$ of the sequences $\tau_k=(t_1,\dots,t_k)$, $\tau_0=\emptyset$, $t_0=0$. This decomposition is associated with a W*-representation $\mathcal{N}_t\simeq\bigoplus_{k=0}^{\infty}\mathcal{M}^{\otimes k}\otimes\mathcal{L}^{\infty}(\Omega_k^t)$ of the algebras $\mathcal{N}_t=m(\mathcal{A}\otimes\mathcal{L}^{\infty}(0,t])$ on the Hilbert spaces $\mathcal{H}_t=\bigoplus_{k=0}^{\infty}\mathcal{F}^{\otimes k}\otimes\mathcal{L}^2(\Omega_k^t)$, giving the construction of the quasi-Poissonian quantum process as the birth process simultaneously of n_i, $i=1,2,\dots$ quantums independently at the corresponding moments t_i, $i=1,2,\dots$ in the states $\varphi_{n_i}/\langle\varphi_{n_i},I^{\otimes n_i}\rangle$ with probabilities $\langle\varphi_{n_i},I^{\otimes n_i}\rangle/\langle\varphi,I^{\otimes}\rangle$ and Poissonian distribution $\mu^t(d\tau_k)=e^{-ct}\langle\varphi,I^{\otimes}\rangle dt_1\cdots dt_k$ of the birth moments $\tau_k\in\Omega^t$, $k=0,1,\dots$, on $\Omega^t=\sum_{k=0}^{\infty}\Omega_k^t$, where Ω_0^t is a one-point space $\{\emptyset\}$ with $d\tau_0=1$.

2. Quantum point processes, multiquantum semigroups and Master equations

Now let us consider an increasing family $\{\mathcal{N}_t\}$ of W*-algebras $\mathcal{N}_t, t \in \mathbb{R}^+$,

$$\mathcal{N}_{r+s} = \mathcal{N}_r \otimes \mathcal{N}_s , \qquad \forall r,s \in \mathbb{R}^+, \quad (2.1)$$

$\mathcal{N}_0 = \mathbb{C}$, with a family $\{\omega^t\}$ of product states $\omega^t : \mathcal{N}^t \to \mathbb{C}$, $\omega^0 = 1$,

$$\omega^{r+s} = \omega^r \otimes \omega^s , \qquad \forall r,s \in \mathbb{R}^+,$$

which is treated as quantum noise, or thermostat.

We define a model of the quantum point process over a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{E})$ as a quantum stochastic process in the sense [8] over multiquantum W*-algebra $\mathcal{M} = m(\mathcal{A})$, described by a family $\{\pi_t\}$ of faithful W*-representations $\pi_t : \mathcal{M} \to \mathcal{M} \otimes \mathcal{N}_t$. We shall consider only quantum point processes, satisfying the stationary Markovian condition

$$\mathcal{E}_t^{t+s} \circ \pi_{t+s} = \pi_t \circ \mathcal{E}_0^s \circ \pi_s , \qquad \forall t,s \in \mathbb{R}^+, \quad (2.2)$$

where $\mathcal{E}_t^{t+s} : \mathcal{M} \otimes \mathcal{N}_{t+s} \to \mathcal{M} \otimes \mathcal{N}_t$ are conditional states, uniquely defined by the states ω^s as linear $\mathcal{M} \otimes \mathcal{N}_t$ -valued functionals, satisfying the condition

$$\mathcal{E}_t^{t+s}(\hat{A} \otimes \hat{B}) = \hat{A} \langle \omega^s, \hat{B} \rangle , \qquad \forall \hat{A} \in \mathcal{M} \otimes \mathcal{N}_t, \hat{B} \in \mathcal{N}_s . \quad (2.3)$$

In this case a one parameter semigroup $\{\hat{\Pi}(t)\}$

$$\hat{\Pi}(r+s) = \hat{\Pi}(r) \circ \hat{\Pi}(s) , \qquad \forall r,s \in \mathbb{R}^+, \quad (2.4)$$

$\hat{\Pi}(0) = I^{\otimes}$ of w*-continuous $\mathcal{M}$-valued states $\hat{\Pi}(t) = \bigoplus_{n=0}^{\infty} \Pi^{(n)}(t)$ is defined on $\mathcal{M} = m(\mathcal{A})$ as the completely positive w*-continuous contractive (normalized) maps

$$\hat{\Pi}(t) = \varepsilon_0^t \circ \pi_t : \mathcal{M} \to \mathcal{M} . \tag{2.5}$$

The simplest quantum point process of this kind is the quasi-Poissonian birth process, which is defined for the family of multiquantum algebras $\mathcal{N}_t = m(\mathcal{B}_t)$ over the von Neumann algebras $\mathcal{B}_t = \mathcal{A} \otimes \mathcal{L}^\infty(0,t]$ with quasi-Poissonian family of states $\{\omega^t\}$ by the family of representations $\pi_t = m(\iota \oplus \iota_t) \equiv \hat{\iota}_t$,

$$\hat{\iota}_t(X^\otimes) = X^\otimes \otimes \iota_t(X)^\otimes , \qquad \forall X \in \mathcal{A}^1, \tag{2.6}$$

where $\iota(A) = A$, and $\iota_t : \mathcal{A} \to \mathcal{B}_t$ is the canonical representation $\iota_t(X) = X \otimes \chi(t)$, $\chi(t) = 1_{(0,t]}$ of $\mathcal{A}$ in $\mathcal{A} \otimes \mathcal{L}^\infty(0,t]$. Representing the W*-algebra $m(\mathcal{B}_{t+s})$ as the W*-tensor product $m(\mathcal{B}_t) \otimes m(\mathcal{B}_s)$ and $\iota_{t+s}(X)^\otimes$ as $\iota_t(X)^\otimes \otimes \iota_s(X)^\otimes$, we obtain the stationary Markovian condition (4.2) for the generating elements $X^\otimes$, $X \in \mathcal{A}^1$:

$$\varepsilon_t^{t+s}(X^\otimes \otimes \iota_{t+s}(X)^\otimes) = X^\otimes \otimes \iota_t(X)^\otimes \langle \omega^s, \iota_s(X)^\otimes \rangle \tag{2.7}$$

with $\langle \hat{\Pi}(t), X^\otimes \rangle = \langle \omega^t, \iota_t(X)^\otimes \rangle X^\otimes = p(t,X) X^\otimes$.

So the semigroup $\hat{\Pi}$ is described by the family $\{\hat{P}(t)\}$ of generating maps $\hat{P}(t,X) = \langle \hat{\Pi}(t), X^\otimes \rangle$ of the form $\hat{P}(t,X) = \oplus P^{(n)}(t,X)$,

$$P^{(n)}(t,X) = \langle \Pi^{(n)}(t), X^\otimes \rangle = p(t,X) X^{\otimes n}, \tag{2.8}$$

where $p(t,X)$ is a multiplicative semigroup

$$p(r,X)\, p(s,X) = p(r+s, X) \tag{2.9}$$

of generating functionals

$$p(t,X) = \langle \omega^t, \iota_t(X)^\otimes \rangle = w^t(\iota_t(X)) \tag{2.10}$$

for states $\rho(t)=\omega^t \circ m(\iota_t)$ on $\mathcal{A}$ with $\iota_t(X)=X\otimes\mathcal{X}(t)$. These states form a convolution semigroup

$$\rho_m(r+s)=\sum_{n=0}^{m} \rho_n(r) \circledS \rho_{m-n}(s) \equiv (\rho(r) \circledast \rho(s))_n \quad , \tag{2.11}$$

corresponding to (2.9) with $\circledast$ defined by symmetrical tensor products $\circledS$ of $\rho_n(r)$ and $\rho_k(s)$, $n+k=m$. So the Markovian maps (2.5) are defined as conjugated to the convolutions of $\rho(s)$ with some state $\sigma \in \mathcal{M}_*$:

$$\sigma \circ \hat{\Pi}(t) = \sigma \circledast \rho(t) = (\sigma \otimes \omega^t)\circ \hat{\iota}_t \quad , \tag{2.12}$$

which have zero matrix elements $\Pi_m^{(n)}(t): \mathcal{A}^{(m)} \to \mathcal{A}^{(n)}$ if $m<n$ and $\sigma_n \Pi_m^{(n)}(t)=(\sigma_n \otimes \omega_{m-n}^t)\circ \iota_t$ describes at $m>n$ the transition of an n-quantum state $\sigma_n \in \mathcal{A}_*^{(n)}$ into an m-quantum state, corresponding to the birth of m-n quantums in the state $\rho_{m-n}(t)$ for the time t.

If the quasi-Poissonian family $\{\omega^t\}$ is continuous in terms of the generating functional w^t of the form (1.4), then the semigroup $\{\hat{\Pi}(t)\}$ has a bounded generator $\hat{\Lambda}=d\hat{\Pi}(t)/dt\mid t=0$, which is conjugated to the convolution generator

$$\sigma \circ \Lambda = \sigma \circledast \varphi - c\sigma \quad , \tag{2.13}$$

corresponding to the multiplication generator

$$s(X)\,\ell(X) = s(X)f(X) - c\,s(X) \tag{2.14}$$

of the semigroup (5.7) for $s(X)=\langle \sigma, X^{\otimes}\rangle$ with $\ell(X)=$ $= d\rho(t)/dt\mid t=0 = f(X)-c$.

In accordance with (1.9) the generator (2.13) uniquely defines the Markovian semigroup $\hat{\Pi}(t)=exp\{t\hat{\Lambda}\}$ as a convolution on the semigroup $\{\rho(t)\}$ by the formula

$$\rho(t) = e^{-ct}\sum_{k=0}^{\infty}\frac{t^k}{k!}\varphi^{\circledast k} \equiv e_{\circledast}^{t(\varphi-c)} \quad . \tag{2.15}$$

Now let us consider the question of the construction of an arbitrary Markovian semigroup as the family $\{\hat{\Pi}(t)\}$ of linear w*-continuous completely positive contractions $\hat{\Pi}(t): \mathcal{M}\to\mathcal{M}$, having a given generator

$$\hat{\Lambda} = \oplus \Lambda^{(n)}, \quad \Lambda^{(n)} = d\Pi^{(n)}(t)/dt\mid t=0 \quad , \tag{2.16}$$

described by linear w*-continuous forms $\Lambda^{(n)}: \mathcal{A}^{(n)} \to \mathcal{M}$. Note, that the direct sum $\hat{\Lambda} = \bigoplus_{n=0}^{\infty} \Lambda^{(n)}$ is w*-continuous iff the family $\{\Lambda^{(n)}\}$ is uniformly bounded, which we do not suppose. Such a generator is described by an (unbounded) generating map

$$\hat{L}(X) = \bigoplus_{n=0}^{\infty} L^{(n)}(X), \quad L^{(n)}(X) = dP^{(n)}(t,X)/dt\,|_{t=0} \tag{2.17}$$

with w*-analytical in $\mathcal{A}^1_*$-maps $L^{(n)}: \mathcal{A}^1 \to \mathcal{M}$. In accordance with the theorem II.1 and $P^{(n)}(0,X) = X^{\otimes n}$ these maps must have the form

$$L^{(n)}(X) = F^{(n)}(X) - A^{(n)*}X^{\otimes n} - X^{\otimes n}A^{(n)}, \tag{2.18}$$

where $F^{(n)}: \mathcal{A}^1 \to \mathcal{A}^{(n)}$ is a w*-analytical P.D. map, and $A^{(n)} \in \mathcal{A}^{(n)}$. Now we formulate the existence theorem for the Markovian semigroups, giving an extension of the Lindblad theorem [3] on the unbounded generators of such kind, satisfying the conditions of dissipativity

$$F^{(n)}(I) \leq A^{(n)} + A^{(n)*}, \qquad n = 0, 1, \dots .$$

Theorem II.2. Let $L^{(n)}$, $n=0,1,\dots$ be a family of w*-analytical C.P.D. maps $\mathcal{A}^1 \to \mathcal{A}^{(n)}$, satisfying the dissipativity condition $L^{(n)}(I) \leq 0$, and $L^* = L$. Then there exists a w*-continuous semigroup of normal C.P. contractive maps $\hat{\Pi}(t) = \oplus \Pi^{(n)}(t)$ on $\mathcal{M} = \oplus \mathcal{A}^{(n)}$, having the generator (2.16) with $\langle \Lambda^{(n)}, X^{\otimes} \rangle = L^{(n)}(X)$. The generating maps $P^{(n)}(t,X) = \langle \Pi^{(n)}(t), X^{\otimes} \rangle$, n = 0,1, ... are given by the w*-converging series

$$P^{(n)}(t,X) = \sum_{k=0}^{\infty} \int_{t_{k-1}}^{t} \dots \int_{t_0}^{t} M_k^{(n)t}(t_1, \dots, t_k, X)\, dt_1 \cdots dt_k \tag{2.19}$$

where $t_0 = 0$, $M_k^{(n)t}: \Omega_k^t \times \mathcal{A}^1 - \mathcal{A}^{(n)}$ is defined as a function on $\tau_k = (t_1, \dots, t_k)$,

$$t_1 < \dots < t_k, \quad \Omega_k^t = \{\tau_k \mid 0 < t_1 < \dots < t_k \leq t\},$$

and $X \in \mathcal{A}^1$, defined by the inverse-time recurrency

$$M_{k+1}^{(n)t}(t_0,\tau_k,X)=e^{-t_0A^{(n)*}}\langle\Phi^{(n)},\hat{M}_k^{t-t_0}(\tau_k-t_0,X)\rangle e^{-t_0A^{(n)}} \tag{2.20}$$

with $\tau_{k+1}=(t_0,\dots,t_k)$, $\hat{M}_k^t=\bigoplus_{n=0}^{\infty}M_k^{(n)t}$, $\hat{M}_0^t=e^{-t\hat{A}^*}X^{\otimes}e^{-t\hat{A}}$, $\hat{A}=\bigoplus_{n=0}^{\infty}A^{(n)}$ and $\Phi^{(n)}$ defined by representations (2.18) with $F^{(n)}(X)=\langle\Phi^{(n)},X^{\otimes}\rangle$.

Proof. If $L^{(n)}:\mathcal{A}^1\to\mathcal{A}^{(n)}$ are w*-analytical C.P.D. dissipative *-maps, then they have the form (2.17) with

$$F^{(n)}(X)=\sum_{m=0}^{\infty}\langle\Phi_m^{(n)},X^{\otimes m}\rangle,\qquad F^{(n)}(I)\leq A^{(n)}+A^{(n)*}.$$

We have to find a solution of the semigroup generating equation $d\hat{P}(t)/dt=$ $=\langle\hat{\Lambda},\hat{P}(t)\rangle$, $\hat{P}(0)=X^{\otimes}$, or

$$\frac{d}{dt}P^{(n)}(t)+A^{(n)*}P^{(n)}(t)+P^{(n)}(t)A^{(n)}=\sum_{m=0}^{\infty}\langle\Phi_m^{(n)},P^{(m)}(t)\rangle \tag{2.21}$$

with the boundary conditions $P^{(n)}(0)=X^{\otimes n}$, n = 0,1,... at least for all positive $X\in\mathcal{A}^1$, defining the matrix elements $\Pi_m^{(n)}(t)$ of the semigroup $\hat{\Pi}(t)$ as derivations of the w*-analytical maps $P^{(n)}(t,X)$. Let us write the system (2.21) with these boundary conditions in the integral form

$$P^{(n)}(t)=e^{-tA^{(n)*}}X^{\otimes n}e^{-tA^{(n)}}+\int_0^t e^{-sA^{(n)*}}\langle\Phi^{(n)},\hat{P}(t-s)\rangle e^{-sA^{(n)}}ds, \tag{2.22}$$

and find the solution $P_i^{(n)}(t)$ of the corresponding recurrent system

$$P_{i+1}^{(n)}(t)=e^{-tA^{(n)*}}X^{\otimes n}e^{-tA^{(n)}}+\int_0^t e^{-sA^{(n)*}}\langle\Phi^{(n)},\hat{P}_i(t-s)\rangle e^{-sA^{(n)}}ds, \tag{2.23}$$

with $P_0^{(n)}(t)=e^{-tA^{(n)*}}X^{\otimes n}e^{-tA^{(n)}}$ as i-th sum of (2.19):

$$P_i^{(n)}(t,X)=\sum_{k\leq i}\int_{t_{k-1}}^{t}\cdots\int_{t_0}^{t}M_k^{(n)t}(t_1,\ldots,t_k,X)\,dt_1\cdots dt_k\,,\ t_0=0. \quad (2.24)$$

As the maps $X\mapsto M_k^{(n)t}(t_1,\ldots,t_k,X)$ defined in (5.20) are completely positive, $P_{i+1}^{(n)}(t)\geq P_i^{(n)}(t)\geq 0$ for all positive $X\in\mathcal{A}^1$. Now we shall prove that $P_i^{(n)}(t)\leq I^{\otimes n}$ if $X\leq I$, i.e. $Q_i^{(n)}(t)=I^{\otimes n}-P_i^{(n)}(t)\geq 0$ for all i = 0,1, ... Indeed, $Q_i^{(n)}(t)$ satisfies the recurrency

$$Q_{i+1}^{(n)}(t)=I^{\otimes n}-e^{-tA^{(n)*}}X^{\otimes n}e^{-tA^{(n)}}-$$

$$-\int_0^t e^{-sA^{(n)*}}[\langle\Phi^{(n)},I^{\otimes}\rangle-\langle\Phi^{(n)},\hat{Q}_i(t-s)\rangle]\,e^{-sA^{(n)}}ds\,,$$

and due to dissipativity $\langle\Phi^{(n)},I^{\otimes}\rangle\leq A^{(n)}+A^{(n)*}$,

$$Q_{i+1}^{(n)}(t)\geq e^{-tA^{(n)*}}(I^{\otimes n}-X^{\otimes n})\,e^{-tA^{(n)}}+$$

$$+\int_0^t e^{-sA^{(n)*}}\langle\Phi,\hat{Q}_i(t-s)\rangle e^{-sA^{(n)}}ds$$

as

$$I^{\otimes n}-e^{-tA^{(n)*}}X^{\otimes n}e^{-tA^{(n)}}-\int_0^t e^{-sA^{(n)*}}(A^{(n)}+A^{(n)*})e^{-sA^{(n)}}ds=$$

$$=I^{\otimes n}-e^{-tA^{(n)*}}X^{\otimes n}e^{-tA^{(n)}}-e^{-tA^{(n)*}}e^{-tA^{(n)}}-I^{\otimes n}=$$

$$=e^{-tA^{(n)*}}(I^{\otimes n}-X^{\otimes n})\,e^{-tA^{(n)}}.$$

So, $Q_0^{(n)}(t) \geqslant 0$, if $X \leqslant I$, and $Q_{i+1}^{(n)}(t) \geqslant 0$, if $Q_i^{(n)}(t) \geqslant 0$. Hence the increasing bounded sequence $0 \leqslant P_i^{(n)}(t) \leqslant I^{\otimes n}$ has a w*-limit $0 \leqslant P_\infty^{(n)}(t) \leqslant I^{\otimes n}$, which is the w*-convergent series (2.19). This limit gives a solution of the integral equation (2.22) because of the increase in (2.23)

$$P_{i+1}^{(n)}(t) \leqslant e^{-tA^{(n)*}} X^{\otimes n} e^{-tA^{(n)}} + \int_0^t e^{-sA^{(n)*}} \langle \Phi^{(n)}, \hat{P}_\infty(t-s) \rangle e^{-sA^{(n)}} ds,$$

$$P_\infty^{(n)}(t) \geqslant e^{-tA^{(n)*}} X^{\otimes n} e^{-tA^{(n)}} + \int_0^t e^{-sA^{(n)*}} \langle \Phi^{(n)}, \hat{P}_i(t-s) \rangle e^{-sA^{(n)}} ds,$$

and, hence,

$$P_\infty^{(n)}(t) \leqslant e^{-tA^{(n)*}} X^{\otimes n} e^{-tA^{(n)}} + \int_0^t e^{-sA^{(n)*}} \langle \Phi^{(n)}, \hat{P}_\infty(t-s) \rangle e^{-sA^{(n)}} ds \leqslant P_\infty^{(n)}(t).$$

Such a solution $P_\infty^{(n)}(t) = P^{(n)}(t, X)$ is, obviously, w*-analytical on $X \in \mathcal{A}^1$ and bounded by

$$P^{(n)}(t, I) = \sum_{k=0}^{\infty} \int_{t_{n-1}}^{t} \cdots \int_{t_0}^{t} M_k^{(n)t}(t_1, \dots, t_n, I) dt_1 \cdots dt_n = I^{\otimes n} - Q_\infty^{(n)}(t) \quad (2.25)$$

<u>Corollary</u> II.2. <u>The Markovian semigroup</u> $\{\hat{\Pi}(t)\}$, <u>defined by the family</u> $P^{(n)}(t, X)$ <u>of generating maps</u> (2.19) <u>gives a weak solution</u> $\sigma(t) = \sigma \circ \hat{\Pi}(t)$ <u>of the Cauchy problem</u> $d\sigma(t)/dt = \sigma(t) \circ \hat{\Lambda}$, $\sigma(0) = \sigma$ <u>for the multiquantum system of Master equations</u>

$$\frac{d}{dt} \sigma_n(t) + A^{(n)} \sigma_n(t) + \sigma_n(t) A^{(n)*} = \sum_{m=0}^{\infty} \sigma_m(t) \circ \Phi_n^{(m)}, \quad (2.26)$$

describing the time-evolution of a multiquantum state $\sigma \in \mathcal{M}_*^+$. This solution is defined by the generating functionals

$$s(t,X)=\sum_{n=0}^{\infty} <\sigma_n, P^{(n)}(t,X)> \equiv <\sigma, \hat{P}(t,X)>, \quad (2.27)$$

where $\hat{A}\sigma = [A^{(n)}\sigma_n]$, $\sigma\hat{A}^* = [\sigma_n A^{(n)*}]$

are defined as dual to left and right multiplications in $\mathcal{A}^{(n)}$:

$$<A^{(n)}\sigma_n, A^{(n)*}> = <\sigma_n, A^{(n)*}A^{(n)}> = <\sigma_n A^{(n)*}, A^{(n)}>.$$

Note, that this solution may not preserve the probability even if $F^{(n)}(I) = A^{(n)} + A^{(n)*}$ for all n in the case of unbounded $\hat{F}(I) = \bigoplus_{n=0}^{\infty} F^{(n)}(I)$.

3. Quasi-free quantum point processes, branching semigroups and nonlinear equations.

The quasifree quantum point processes describe a stochastic evolution of a multiquantum system $\mathcal{M} = m(\mathcal{A})$ without interaction of its particles. An example of such a quantum Markovian process is the quasi-Poissonian birth process defined in the previous section, which is described by a one parameter multiplicative semigroup $\{p(t)\}$ of state generating functionals $p(t):\mathcal{A}^1 \to \mathbb{C}$. Now we shall consider a current semigroup $\{p(t), T(t)\}$ defined as a semi-direct product

$$[p(r), T(r)] \circ [p(s), T(s)] = [(p(r) \circ T(s))p(s), T(r) \circ T(s)]$$

of the family $\{p(t)\}$ with a nonlinear semigroup $\{T(t)\}$ of maps $T(t): \mathcal{A}^1 \to \mathcal{A}^1$, called a branching semigroup in the case $p(t)=1$. We shall show that every multiquantum quasi-free (branching) Markovian semigroup $\{\Pi(t)\}$ over $\mathcal{A}$ is generated by such a current (branching) semigroup, which satisfies the following conditions. A family $\{p(t), T(t)\}$ of w*-analytical maps $p(t): \mathcal{A}^1 \to \mathbb{C}$, $T(t): \mathcal{A}^1 \to \mathcal{A}^1$, $t \in \mathbb{R}^+$

is called a <u>Markovian current semigroup</u>, if

0)
$$p(r+s,X)=p(r,T(s,X))p(s,X) \qquad X\in\mathcal{A}^1$$
$$T(r+s,X)=T(r,T(s,X)) \qquad r,s\in\mathbb{R}^+$$

with $p(o,X)=1$, $T(o,X)=X$ for all $X\in\mathcal{A}^1$.

1) $p(t,I)\leqslant 1$, $T(t,I)\leqslant I$ for all $t\in\mathbb{R}^+$ (or $p(t,I)=1$, $T(t,I)=I$ if they are normalized)

2) for all n = 0,1, ... , the maps $P^{(n)}(t):\mathcal{A}^1\to\mathcal{A}^{(n)}$,

$$P^{(n)}(t,X)=T(t,X)^{\otimes n}\cdot p(t,X)$$

are positive definite (P.D.) in the sense (i) of theorem I.2 for $\mathcal{B}=\mathcal{A}^{(n)}$. Obviously, $p(t)$ is a P.D. functional for all $t\in\mathbb{R}_+$, as it follows from 2) at n = 0 and $T(t)$ is also P.D., in the purely branching case $p(t)=1$. Note, that in the case of P.D. p and T the condition 2) is satisfied for all n.

Let us define the quasifree quantum point Markovian process by the semigroup $\{\hat{\Pi}(t)\}$ of linear w*-continuous completely positive contractions $\hat{\Pi}(t)=\Theta(t)^{\circledast}\circledast\rho(t)$ with $\Pi^{(n)}(t)$ being defined as convolutions

$$\Pi^{(n)}(t)=\Theta(t)^{\circledast n}\circledast\rho(t), \qquad n= \tag{3.1}$$

where $\{\rho(t),\Theta(t)\}$ is a family of multiquantum states $\rho(t)=$ $=\Pi^{(0)}(t)$ $(\Theta^{\circledast 0}=1)$, describing the quantum birth process of particles, and linear w*-continuous branching maps $\Theta(t):\mathcal{M}\to\mathcal{A}$, $\Theta(t)=[\Theta_n(t)]$, $\Theta_n(t):\mathcal{A}^{(n)}\to\mathcal{A}$, $n=0,1,\dots$,
describing the death $\sigma_1\in\mathcal{A}_*^+\mapsto\sigma_1\circ\Theta_0(t)=\langle\sigma_1,\Theta_0(t)\rangle$ the time evolution

$\sigma_1 \in \mathcal{A}_*^+ \mapsto \sigma_1 \circ \Theta_1(t)$, and the spliting $\sigma_1 \in \mathcal{A}_*^+ \mapsto \sigma_1 \Theta_n(t)$, $n > 1$ of each particle in a state σ_1 into n particles for a time $t > 0$. The convolution (3.1) corresponds to the independence of the quantum transitions from an n-quantum state $\sigma_n \in \mathcal{A}_*^{(n)}$ into an m-quantum one which are described by the matrix elements of $\Pi^{(n)}(t) = [\Pi_m^{(n)}(t)]$,

$$\Pi_m^{(n)}(t) = \sum_{k=0}^{m} \Theta(t)^{\circledast n} \circledS \rho_{m-k}(t), \qquad t \in \mathbb{R}^+ \tag{3.2}$$

where $\circledS$ is the symbol of the symmetrical tensor product of birth quantum state $\rho_{m-k}(t)$ and a state $\sigma_n \in \mathcal{A}_*^{(n)}$, transformed by independent branchings

$$\Theta_k(t)^{\circledast n} = \sum_{k_1+\ldots+k_n=k} \Theta_{k_1}(t) \otimes \ldots \otimes \Theta_{k_n}(t)$$

of each quantum into the k_i quantums with total number $\sum k_i = k$ corresponding to state $\sigma_n \circ \Theta_k(t)^{\circledast n}$. Note that in purely branching case, corresponding to vacuum $\rho_n = \delta_n^0$, $\Pi^{(n)}(t) = \Theta(t)^{\circledast n}$, where $\Theta(t) = \Pi^{(1)}(t)$ is a w*-continuous completely positive contraction $\mathcal{M} \to \mathcal{A}$.

The semigroup condition (2.4) defines for the family $\{\rho(t), \Theta(t)\}$ a <u>current convolution semigroup</u> structure

$$\rho(r+s) = \sum_{n=0}^{\infty} \rho_n(r) \circ \Theta(s)^{\circledast n} \circledast \rho(s) \equiv \rho(r) \circ \Theta(s)^{\circledast} \circledast \rho(s),$$

$$\Theta(r+s) = \sum_{n=0}^{\infty} \Theta_n(r) \circ \Theta(s)^{\circledast n} \equiv \Theta(r) \circ \Theta(s)^{\circledast}, \tag{3.3}$$

with $\rho_n(0) = \delta_n^0$, $\Theta_n(0) = \delta_n^1 Id$. This semigroup obeys a generator $\hat{\Lambda} = \oplus \Lambda^{(n)}$,

$$\frac{d}{dt}\,\sigma_n \circ \Pi^{(n)}(0) = d[\sigma_n \circ \Theta(t)^{\circledast n} \circledast \rho(t)]/dt\,\Big|_{t=0} =$$

$$= n\,\sigma_n \circ \left(I^{(n-1)}_{n-1} \circledast \frac{d}{dt}\,\Theta(0)\right) + \sigma_n \circledast \frac{d}{dt}\,\rho(0) = \sigma_n \circ \Lambda^{(n)}$$

for all $\sigma_n \in \mathcal{A}^{(n)}_*$, if-f the derivatives

$$\Gamma = d\Theta(t)/dt\;|t=0\;, \qquad \lambda = d\rho(t)/dt\,|t=0 \tag{3.4}$$

are defined as linear w*-continuous forms $\Gamma : \mathcal{M} \to \mathcal{A}$, $\lambda : \mathcal{M} \to \mathbb{C}$. In this case the matrix elements $\Lambda^{(n)}_m : \mathcal{A}^{(m)} \to \mathcal{A}^{(n)}$ of $\Lambda^{(n)} = [\Lambda^{(n)}_m]$ are defined as conditionally completely positive dissipative forms, which are conjugated to the infinitesimal transitions $\sigma_n \mapsto \sigma_n \Lambda^{(n)}_m$ of n-quantum states $\sigma_n \in \mathcal{A}^{(n)}_*$ into m-quantum

$$\sigma_n \circ \Lambda^{(n)}_m = 0\,, \quad m < n-1\;; \qquad \sigma_n \circ \Lambda^{(n)}_{n-1} = n\,\sigma_n \circ (I^{(n-1)}_{n-1} \circledS \Gamma_0)\,,$$

$$\sigma_n \circ \Lambda^{(n)}_m = n\,\sigma_n \circ (I^{(n-1)}_{n-1} \circledS \Gamma_{m-n+1}) + \sigma_n \circledS \lambda_{m-n}\,, \quad m \geq n\,, \tag{3.5}$$

where $[\Gamma_n] = \Gamma$, $[\lambda_n] = \lambda$, $I^{(n)}_n = Id$ - the identity map $\mathcal{A}^{(n)} \to \mathcal{A}^{(n)}$,

$$n\,(I^{(n-1)}_{n-1} \circledS \Gamma_{m-n+1}) = \sum_{k=1}^{n} I^{(n-k)}_{n-k} \otimes \Gamma_{m-n+1} \otimes I^{(k-1)}_{k-1}$$

are the linear forms, corresponding to the velocity of transitions $\sigma_n \to \sigma_m$ due to the independent branching of each quantum into m-n+1 quantums, and corresponds to the velocity of transitions $\sigma_n \to \sigma_m$ due to simultaneous birth of m-n quantums. We shall call the direct sum $\hat{\Lambda}$ of the forms

$$\Lambda^{(n)} = n\left(Id^{(n-1)} \otimes \Gamma\right) + Id^{(n)} \otimes \lambda\,, \qquad Id^{(n)} = \bigoplus_{m=0}^{\infty} I_m^{(n)} \delta_n^m \quad (3.6)$$

defined by a pair (λ, Γ) thus: the quasi-free point infinitesimal form is conditionally completely positive and dissipative for all n = 0,1, ... The corresponding pairs (λ, Γ) form a cone called the <u>current convolution semialgebra</u>. Of course $\lambda = \Lambda^{(0)} = \lambda^*$ is a conditionally positive and dissipative form $\lambda \in \mathcal{M}_*$, and if $\Gamma = \Gamma^*$ is also conditionally completely positive and dissipative, as it takes place in a purely branching case $\lambda = 0$, then $\oplus \Lambda^{(n)}$ is a quasifree point infinitesimal form. Note that in spite of the boundedness of each the form $\Lambda^{(n)} = [\Lambda_m^{(n)}]$:

$$\| \Lambda^{(n)} \| \leq \| \lambda \| + n \| \Gamma \|$$

the generator $\hat{\Lambda} = \bigoplus_{n=0}^{\infty} \Lambda^{(n)}$ of the quasifree quantum point semigroup is typically unbounded, if $\Gamma \neq 0$.

Let us consider the generating maps

$$p(t, X) = \langle \rho(t), X^{\otimes} \rangle\,, \qquad T(t, X) = \langle \Theta(t), X^{\otimes} \rangle \quad (3.7)$$

for a convolution current semigroup $\{\rho(t), \Theta(t)\}$, and

$$\ell(X) = \langle \lambda, X^{\otimes} \rangle\,, \qquad G(X) = \langle \Gamma, X^{\otimes} \rangle \quad (3.8)$$

for its generator, defined as w*-analytical maps $p(t), \ell : \mathcal{A}^1 \to \mathbb{C}$ and $T(t), G : \mathcal{A}^1 \to \mathcal{A}$. We shall say that a pair (ℓ, G) is completely dissipative, if $\ell^* = \ell$, $G^* = G$,

1') $\ell(I) \leq 0\,, \quad G(I) \leq 0 \qquad (\ell(I) = 0\,, \quad G(I) = 0)$

2') for all n = 0,1, ... the maps $L^{(n)} : \mathcal{A}^1 \to \mathcal{A}^{(n)}$,

$$L^{(n)}(X) = \ell(X)\, X^{\otimes n} + \sum_{k=1}^{n} X^{\otimes(n-k)} \otimes G(X) \otimes X^{\otimes(k-1)}$$

are C.P.D. with respect to $\mu(X) = X^{\otimes n}$, or $\mu = id^{\otimes n}$ in the sense (i) of Theorem II.1 for $\mathcal{B} = \mathcal{A}^{(n)}$, $\mathcal{G} = \mathcal{E}^{\otimes n}$. Such pairs (ℓ, G) form a cone called a current semialgebra, or branching semialgebra of w*-analytical C.P.D. dissipations $G = L^{(1)} : \mathcal{A}^1 \to \mathcal{A}$, if $\ell = 0$.

Of course, $\ell = L^0$ is a C.P.D. functional:

$$\ell(X) = f(X) - c \quad \text{with P.D. } f : \mathcal{A}^1 \to \mathbb{C} \text{ and } c \geq f(I), \tag{3.9}$$

and, if G is a C.P.D. map with respect to $\mu(X) = X$, then

$$G(X) = K(X) - B^*X - XB \text{ with P.D. } K : \mathcal{A}^1 \to \mathcal{A} \text{ and } B + B^* \geq K(I). \tag{3.10}$$

Theorem II.3. A family $\{P^{(n)}(t)\}$, $t \in \mathbb{R}^+$ of maps $P^{(n)}(t) : \mathcal{A}^1 \to \mathcal{A}^{(n)}$, n = 0,1, ... is a generating family for a quasifree point Markovian semigroup, iff it has the form $T(t)^{\otimes n} \cdot p(t)$, where $(p(t), T(t))$, $t \in \mathbb{R}^+$ is a current Markovian semigroup, defined by the conditions 0) – 4). A family $\{L^{(n)}\}$ of maps $L^{(n)} : \mathcal{A}^1 \to \mathcal{A}^{(n)}$, n = 0,1, ... is a generating family for a quasifree point infinitesimal form, if it has the form $id^{\otimes n}\ell + n\, id^{\otimes(n-1)} \circledS G$, where (ℓ, G) is a current completely dissipative pair, defined by conditions 1'), 2'). If G is a w*-analytical C.P.D. dissipative map $\mathcal{A}^1 \to \mathcal{A}$, then there exists a w*-continuous semigroup $\{T(t)\}$ of w*-analytical contractions $\mathcal{A}^1 \to \mathcal{A}^1$, having the generator $G = dT(t)/dt \mid t=0$, defining the current Markovian semigroup with $p(t) = \exp\{\int_0^t \ell(T(s))ds\}$ for a w*-analytical C.P.D. dissipative functional $\ell : \mathcal{A}^1 \to \mathbb{C}$.

Proof. The equivalence of (3.1) and (3.6) with corresponding forms in 2) and 2') for generating maps

$$P^{(n)}(X) = \langle \Pi^{(n)}, X^{\otimes} \rangle, \quad L^{(n)}(X) = \langle \Lambda^{(n)}, X^{\otimes} \rangle$$

is a consequence of the definition of convolution (3.2) and (3.5):

$$\sum_{m=0}^{\infty} \langle \Pi_m^{(n)}, X^{\otimes m} \rangle = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \langle \Theta_k^{\otimes n}, X^{\otimes k} \rangle \langle \rho_{m-k}, X^{\otimes (m-k)} \rangle =$$

$$= \sum_{k=0}^{\infty} \sum_{k_1+\ldots+k_n=k} \langle \Theta_{k_1}, X^{\otimes k_1} \rangle \otimes \ldots \otimes \langle \Theta_{k_n}, X^{\otimes k_n} \rangle \langle \rho, X^{\otimes} \rangle =$$

$$= \langle \Theta, X^{\otimes} \rangle^{\otimes n} \langle \rho, X^{\otimes} \rangle = T(X)^{\otimes n}\, p(X) ;$$

$$\sum_{m=0}^{\infty} \langle \Lambda_m^{(n)}, X^{\otimes m} \rangle = n\, X^{\otimes (n-1)} \circledS \sum_{m=n-1}^{\infty} \langle \Gamma_{m-n+1}, X^{\otimes (m-n+1)} \rangle +$$

$$+ \sum_{m=n}^{\infty} \langle \lambda_{m-n}, X^{\otimes m} \rangle = n\, X^{\otimes (n-1)} \circledS G(X) .$$

Then the conditions 1), 2) and 1'), 2') are simply the conditions defining the generating families $P^{(n)}$ and $L^{(n)}$, n = 0,1, ... for Markovian maps and their infinitesimal forms respectively. In the same way we prove the equivalence of current semigroup conditions (0) and (3.3):

$$\sum_{m=0}^{\infty} \langle \rho_m(r+s), X^{\otimes m} \rangle = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \langle \rho(r) \circ \Theta_k(s)^{\otimes}, X^{\otimes k} \rangle \langle \rho_{m-k}(s), X^{\otimes (m-k)} \rangle$$

$$= \sum_{n=0}^{\infty} \langle \rho_n(r), \langle \Theta(s)^{\otimes n}, X^{\otimes} \rangle \rangle \langle \rho(s), X^{\otimes} \rangle = p(r, T(s,X))\, p(s,X),$$

$$\sum_{m=0}^{\infty} \langle \Theta_m(r+s), X^{\otimes m} \rangle = \sum_{m=0}^{\infty} \langle \Theta(r) \circ \Theta_m(s)^{\otimes}, X^{\otimes m} \rangle =$$

$$= \sum_{n=0}^{\infty} \langle \Theta_n(r), \langle \Theta(s), X^{\otimes} \rangle^{\otimes n} \rangle = T(r, T(s,X)) .$$

Now we shall prove the existence of the current Markovian semigroup $(p(t), T(t))$, having the generator of the form (3.9), (3.10). So we have to find the resolving pair $T(t,X)$, $p(t,X)$ for the Cauchy problem

$$dT(t)/dt + B^*T(t) + T(t)B = K(T(t)), \quad T(0)=X,$$
$$dp(t)/dt + c\,p(t) = p(t)f(T(t)), \quad p(0)=1, \tag{3.11}$$

with w*-analytical C.P. maps $K: \mathcal{A}' \to \mathcal{A}$ and $f: \mathcal{A}' \to \mathbb{C}$ and $B \in \mathcal{A}$, $c \in \mathbb{C}$, satisfying dissipativity conditions $B + B^* \geqslant K(I)$, $c \geqslant f(I)$. The first equation is equivalent to the integral nonlinear equation

$$T(t) = e^{-tB^*}Xe^{-tB} + \int_0^t e^{-sB^*}K(T(t-s))e^{-sB}ds, \tag{3.12}$$

which we solve for $0 \leqslant X \leqslant I$ by sequential iterations

$$T_{i+1}(t) = e^{-tB^*}Xe^{-tB} + \int_0^t e^{-sB^*}K(T_i(t-s))e^{-sB}ds, \quad i=0,1,\ldots$$

with $T_0(t) = e^{-tB^*}X\,e^{-tB}$. Due to increasing K $T_1(t) \geqslant T_0(t)$ for all t, and

$$T_{i+1}(t) - T_i(t) = \int_0^t e^{-sB^*}[K(T_i(t-s)) - K(T_{i-1}(t-s))]e^{-sB}ds \geqslant 0$$

if $T_i(t) \geqslant T_{i-1}(t)$ for all t. So $T_i(t)$, i = 0,1, ... is an increasing family $0 \leqslant T_0(t) \leqslant T_1(t) \cdots$. Let us prove the boundness $T_i(t) \leqslant I$ if $X \leqslant I$. The difference $Q_i(t) = I - T_i(t) \leqslant I$ satisfies the recurrence

$$Q_{i+1}(t) = \int_0^t e^{-sB^*} K(I - Q_i(t-s)) e^{-sB} ds \geqslant$$

$$\geqslant I - e^{-tB^*} X e^{-tB} - \int_0^t e^{-sB^*} K(I) e^{-sB} ds \geqslant$$

$$\geqslant I - e^{-tB^*} X e^{-tB} - \int_0^t e^{-sB^*} (B + B^*) e^{-sB} ds =$$

$$= e^{-tB^*} (I - X) e^{-tB} \geqslant 0 ,$$

so the sequence $T_i(t)$ has a w*-limit $T_\infty(t) = T(t, X)$, which is a w*-analytical on $X \in \mathcal{A}^1$ solution of (3.12) for $0 \leqslant X \leqslant I$, using the same arguments as in the linear case considered in theorem (2.2). Due to the analyticity $T(t)$ defines a solution of the problem (3.11) for all $X \in \mathcal{A}^1$ with

$$p(t, X) = \exp\left\{ \int_0^t f(T(s, X)) ds - ct \right\}$$ ▨

Corollary II.3. The nonlinear semigroup $T(t) : \mathcal{A}^1 \to \mathcal{A}^1$, defined by the w*-analytical integral equation (3.12), gives a quasifree solution $P^{(n)}(t) = T(t)^{\otimes n} \cdot p(t)$, n = 0,1, ... of the Markovian system of integral equations (2.22) with $A^{(n)} = n I^{\otimes(n-1)} ⓢ B + I^{\otimes n} c/2$,

$$\varphi_m^{(n)} = n I_{n-1}^{(n-1)} ⓢ K_{m-n+1} + I_n^{(n)} ⓢ \phi_{m-n} ,$$

defined by the corresponding quasifree forms (3.6) with

$$\langle \Gamma, \hat{A} \rangle = \sum_{n=0}^{\infty} \langle K_n , A^{(n)} \rangle - B^* A^{(1)} - A^{(1)} B$$

and $\langle \lambda, \hat{A} \rangle = \sum_{n=0}^{\infty} \langle \varphi_n, A^{(n)} \rangle - c A^{(0)}$. Here $K_n : \mathcal{A}^{(n)} \to \mathcal{A}$ are completely positive operations, with generating map $K(X) = \sum_{n=0}^{\infty} \langle K_n, X^{\otimes n} \rangle$, which are defined as conjugated to the velocity of branching transitions $\mathcal{A}_* \to \mathcal{A}_*^{(n)}$ and the forms $\varphi_n \in \mathcal{A}_*^{(n)}$, $\sum_{n=0}^{\infty} \langle \varphi_n, I^{\otimes n} \rangle \leq c$, corresponding to the velocity of n-quantums birth quasi-Poissonian process. In the particular case of the purely branching process (c = 0 and, hence $\varphi_n = 0$ for all n) we get a branching solution $P^{(n)}(t) = T(t)^{\otimes n}$ and, hence $\Pi^{(n)}(t) = \Theta(t)^{\otimes n}$, n = 1,2, ... for the corresponding system (2.22), defined by branching infinitesimal forms $\Lambda^{(n)} : \hat{A} \in \mathcal{M} \mapsto \mathcal{A}^{(n)}$:

$$\langle \Lambda^{(n)}, \hat{A} \rangle = n\left(\sum_{i=0}^{\infty} \langle I_{n-1}^{(n-1)} \circledS K_{i+1}, A^{(n+i)} \rangle - (I^{\otimes(n-1)} \circledS B^*) A^{(n)} - A^{(n)}(I^{\otimes(n-1)} \circledS B)\right)$$

with the generating maps $L^{(n)} = n I^{\otimes(n-1)} \circledS G$, n = 1,2, ..., defined by w*-analytical C.P.D. dissipation (3.10).

Indeed, if a nonlinear equation $dT/dt = G(T)$ with w*-analytical $G(X) = \langle \Gamma, X^{\otimes} \rangle$ has a solution $T(t)$ for a $T(0) = X$, then the corresponding system of linear equations

$$\frac{d}{dt} P^{(n)} = \sum_{i=0}^{\infty} \langle n I_{n-1}^{(n-1)} \circledS \Gamma_{i+1} + I_n^{(n)} \circledS \lambda_i, P^{(n+i)} \rangle$$

with $\lambda_n \in \mathcal{A}_*^{(n)}$ defined by a w*-analytical functional $\ell(X) = \langle \lambda, X^{\otimes} \rangle$ is satisfied by $P^{(n)}(t) = T(t)^{\otimes n} p(t)$ with $dp(t)/dt = \ell(T(t))$:

$$\frac{d}{dt} P^{(n)}(t) = \sum_{i=0}^{\infty} \langle n I_{n-1}^{(n-1)} \circledS \Gamma_{i+1} + I_n^{(n)} \circledS \lambda_i, T^{\otimes(n+i)} p \rangle =$$

$$= p \sum_{i=0}^{\infty} (n T^{\otimes(n-1)} \circledS G(T) + T^{\otimes n} \ell(T)) = \frac{d}{dt} T^{\otimes n} p .$$

This solution as a function of $X \in \mathcal{A}^1$ defines a weak solution of the Cauchy problem for the corresponding Master equation (2.26), by the generating functional

$$s(t,X) = \sum_{n=0}^{\infty} \langle \sigma_n , T(t,X)^{\otimes n} p(t,X) \rangle = s(T(t,X))p(t,X),$$

where $s(X) = \langle \sigma, X^{\otimes} \rangle$ is an initial multiquantum state generating functional.

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