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CARTAN-PRESERVING AUTOMORPHISMS AND THE WEYL GROUP OF KAC-MOODY ALGEBRAS.

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ABSTRACT

The group $\tilde{N}$ defined as the normalizer of the Cartan subalgebra in the group of all (inner and outer) automorphisms of affine Kac-Moody (KM) algebras is shown to play a fundamental role in the structure of these algebras. It is a (discrete) Galilean group which incorporates the affine and Weyl group structure of the KM algebra and the space-time structure of the bosonic string. It links the Virasoro and KM algebras in a non-trivial way and it plays a key role in the 'vertex' construction.

1. INTRODUCTION

Experience has shown that the automorphism groups of physical systems and equations often transcend in importance the particular systems and equations to which they apply. For example, the three major automorphisms of classical electromagnetism – the Poincaré, conformal and gauge automorphisms – have all become cornerstones of modern physics and apply to a much wider class of systems. For this reason the study of automorphism groups of new systems is important, and in the present paper we wish to study an automorphism group of Kac-Moody (KM) algebras.[1,2] Of course, some of the automorphisms of KM algebras, namely the conformal (Virasoro) automorphisms[3] and, in the case of the string,[4] the Poincaré automorphisms, have already been well-studied, but what we wish to consider here is a different group of automorphisms, namely the group $\tilde{N}$ of all automorphisms (both inner and outer) of the algebra which preserve the Cartan subalgebra. The inner automorphism part of this group is known in connection

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with the affine and Weyl structures of KM algebras, but what we wish to do here is to emphasize its importance for the general structure of KM algebras and string theory, and to explore its properties in detail, in particular to distinguish between its inner and outer parts.

The group of all Cartan-preserving automorphisms is called the normalizer of the Cartan in the group of automorphisms and will be denoted by $\tilde{N}$, the subgroup of inner automorphisms being denoted by N. The action of $\tilde{N}$ on the Cartan is determined by the quotient group $\tilde{W} = \tilde{N}/C$, where C is the centralizer of the Cartan i.e. the group of automorphisms that leave each element of the Cartan separately invariant. (For the KM and Lie algebras C is automatically inner). For ordinary compact simple Lie algebras the groups $W^0 = N^0/C^0$ and $\tilde{W}^0 = \tilde{N}^0/C^0$ are just the Weyl group and the semi-direct product of the outer automorphisms and the Weyl group, respectively.[5] Thus $\tilde{W}^0 = \mathcal{T} \wedge W^0$ where $\mathcal{T}$ is the group of outer automorphisms. It will be seen that in the KM case $W = N/C$ is still the Weyl group but W, $\tilde{W}$, N and $\tilde{N}$ have much richer structures than for ordinary Lie algebras. More precisely, it will be shown in this paper that

(a) The group $\tilde{N}$ has the structure of a Galilean group, whose parameters take discrete values and whose homogeneous (rotation) subgroup is just the corresponding group $\tilde{N}^0$ for the ordinary Lie subalgebra G_0 of the KM algebra. The new, characteristic KM feature is the existence of an acceleration subgroup $\tilde{A}$.

(b) The quotient group $\tilde{W} = \tilde{N}/C$ is isomorphic to a group generated by reflexions of the (Minkowskian) root diagram of the KM algebra. In particular, the inner subgroup $W = N/C$ is isomorphic to the subgroup generated by those Minkowskian reflexions which are in planes orthogonal to the KM roots and is thus identified with the KM Weyl reflexions considered by previous authors.[3]

(c) The elements of the strictly outer part $\tilde{N}/N$ of $\tilde{N}$ permute the highest weight representations of the KM algebra which are defined by the different fundamental representations of G_0 on their vacuum-states.

(d) The group $\tilde{N}$ plays not only the role of an 'internal' Weyl group but, in string theory, it also plays the role of an 'external' space-time group. In particular, the generators of the acceleration subgroup $\tilde{A}$ become the centre-of-mass coordinates of the string, and thus introduce the centre-of-mass coordinates in a natural way, while the group $\tilde{N}$ itself is the subgroup of the Poincaré group which is the little group of a light-like vector in Minkowskii space, a result that explains the Galilean structure of $\tilde{N}$.

(e) The action of $\tilde{N}$ can be extended (uniquely) to include the Virasoro algebra V of the KM algebra and it then links the V and KM algebras in a manner that is reducible but not fully reducible.

(f) The elements of N play a key role in the so-called 'vertex' construction[3)] of the non-abelian elements of the KM algebra.

The properties (a)–(f) clearly demonstrate the importance of the Cartan-preserving automorphism groups for KM algebras.

2. PROPERTIES OF COMPACT SIMPLE LIE ALGEBRAS

The simple, compact, simply-connected Lie groups[5)] will be denoted by $\mathcal{G}_0$ and their Lie algebras

$$[T^a, T^b] = f^{ab}{}_c T^c \ , \quad a, b, c = 1, 2, ..., d \tag{2.1}$$

by G_0. Here the $f^{ab}{}_c$ are real totally anti-symmetric structure constants, uniformly normalized so that (by Schur's lemma) they satisfy

$$f^{ac}{}_d f^{db}{}_c = Q\delta^{ab} \ , \tag{2.2}$$

where Q is a constant that depends only on the group. The Cartan form of the algebra will be written in the usual way as

$$[H^i, E^\alpha] = \alpha^i E^\alpha \ , \quad [E^\alpha, E^{-\alpha}] = \frac{2}{\alpha^2}(\alpha.H) \ , \quad i = 1, ..., l, \tag{2.3}$$

where l is the rank, the H's commute and there are further relations between the E^α's which will not be needed here. From (2.2) the root vectors α evidently satisfy the completeness relation

$$\sum_\alpha \alpha^i \alpha^j = Q\delta^{ij} \ , \tag{2.4}$$

where α^i are the components of α.

The roots are characterised by the property that for any two roots α, β,

$$2(\alpha.\beta)/\beta^2 \in \mathrm{Z} \ , \tag{2.5}$$

where $\mathbf{Z}$ denotes the set of all integers, and this leads one to introduce <u>coroots</u> $\tilde{\alpha}$ defined by

$$\tilde{\alpha} = 2\alpha/\alpha^2 \quad \text{so that} \quad (\tilde{\alpha}.\beta) \in \mathbf{Z} . \tag{2.6}$$

The coroots are obviously parallel to the roots, and since they, in turn, satisfy (2.5) they are the roots of an algebra $\tilde{G}_0$, called the dual algebra. However, the dual algebra is actually the algebra itself (G_0 is self-dual) in all cases except $SO(2n+1)$ and $Sp(2n)$ which are dual to each other.

The roots and coroots generate (by addition and subtraction) infinite lattices called the root and coroot lattices Γ_α and $\Gamma_{\tilde{\alpha}}$ respectively. Furthermore, to each set of roots and coroots there corresponds a dual lattice called the weight and coweight lattice, Γ_w and $\Gamma_{\tilde{w}}$, respectively, and defined as the set of all weights w and coweights $\tilde{w}$ satisfying

$$(\tilde{\alpha}.w) \in \mathbf{Z} \qquad (\alpha.\tilde{w}) \in \mathbf{Z} . \tag{2.7}$$

From (2.5) it is clear that roots and coroots are special cases of weights and coweights respectively, but except for $G_0 = E_8$ the converse is not true so, in general,

$$\Gamma_w \supset \Gamma_\alpha \quad \text{and} \quad \Gamma_{\tilde{w}} \supset \Gamma_{\tilde{\alpha}} . \tag{2.8}$$

Note that α and w scale in the same way, and oppositely to $\tilde{\alpha}$ and $\tilde{w}$. The algebras for which all the roots have the same length are called simply-laced algebras and for the other algebras (for which the roots can have only two distinct lengths) it will be convenient to denote the long and short roots by λ and σ respectively, and to define primitive long and short coweights, $\tilde{w}_{\lambda_i}$ and $\tilde{w}_{\sigma_i}$ dual to the primitive long and short roots, λ_i and σ_i, in the usual way (i.e. $(w_{\lambda_i}.\tilde{\lambda}_j) = \delta_{ij}$, $(w_{\sigma_i}.\tilde{\sigma}_j) = \delta_{ij}$). If the lattices generated by each set of weights and roots are denoted by Γ_{w_λ}, Γ_{w_σ}, Γ_λ, Γ_σ, with analogous definitions of long and short coroots, coweights and their associated lattices, then we have the following identities

$$\begin{aligned} \Gamma_{\tilde{\lambda}} &= \frac{2}{\lambda^2}\Gamma_\lambda \quad \text{and} \quad \Gamma_{\tilde{\sigma}} = \frac{2}{\sigma^2}\Gamma_\sigma \\ \Gamma_{\tilde{w}_\lambda} &= \frac{2}{\lambda^2}\Gamma_{w_\lambda} \quad \text{and} \quad \Gamma_{\tilde{w}_\sigma} = \frac{2}{\sigma^2}\Gamma_{w_\sigma} , \end{aligned} \tag{2.9}$$

from which we obtain the inequalities

$$\frac{\lambda^2}{2}\Gamma_{\tilde{\alpha}} \subseteq \Gamma_\alpha \subseteq \frac{\sigma^2}{2}\Gamma_{\tilde{\alpha}} \quad \text{and} \quad \frac{\lambda^2}{2}\Gamma_{\tilde{w}} \subseteq \Gamma_w \subseteq \frac{\sigma^2}{2}\Gamma_{\tilde{w}} , \tag{2.10}$$

which will be useful later.

As is well known, the positive weights w are the highest weights of the irreducible representations of the algebras, and they fall naturally into congruency classes (the generalisations of the SU(3) triality classes) where two weights w and w' are said to be congruent ($w \sim w'$) if, and only if, they differ by an element of the root-lattice,

$$w \sim w' \iff w - w' \in \Gamma_\alpha . \tag{2.11}$$

The lowest positive weight $w^{(g)}$ in each congruency class will be called a fundamental weight and the representation with $w^{(g)}$ as highest weight a fundamental representation. It is easy to see that the group elements

$$\exp 2\pi i(\tilde{w}.H) \tag{2.12}$$

of $\mathcal{G}_o$ are central, and it can be shown that all elements of the centre are of this kind. Also, since

$$\exp 2\pi i(\tilde{\alpha}.H) = 1 , \tag{2.13}$$

the central elements corresponding to two coweights $\tilde{w}_1$ and $\tilde{w}_2$ are the same if and only if the coweights are congruent (modulo $\tilde{\alpha}$). This shows that the order p of the centre is just the number of congruency classes of the dual group and hence of the group itself. Equation (2.12) also shows that the irreducible representations fall naturally into congruency classes, each characterised by a fundamental representation. Since $p \leq l + 1$ for all groups and $p = l + 1$ only for $SU(l + 1)$, the number of fundamental representations is, in general, less than $l + 1$.

A crucial group for our considerations will be the group $\tilde{N}^0$ of all automorphisms of the Lie algebra G_0 that preserve the Cartan subalgebra. Since $\tilde{N}^0$ is an orthogonal group and can at most permute the roots and since every orthogonal permutation of the roots induces a Cartan-preserving automorphism of the Lie algebra ($\alpha.H \rightarrow \beta.H$, $E^\alpha \rightarrow E^\beta$), it is clear that $\tilde{N}^0$ is isomorphic to the group of orthogonal permutations of the roots. It is not difficult to show that $\tilde{N}^0$ is a semi-direct product of the form

$$\tilde{N}^0 = \mathcal{T} \wedge W^0 , \tag{2.14}$$

where $\mathcal{T}$ is the group $Aut(G_0)/Int(G_0)$ of strictly outer automomorphisms of G_0, and W^0 is the group of Cartan-preserving inner automorphisms. It is well known that $\mathcal{T}$ is non-trivial only for the groups listed in the upper row of (2.15), and that $\mathcal{T} = S_2$ in all cases except $SO(8)$, for which $\mathcal{T} = S_3$, and that the maximal $\mathcal{T}$-invariant subgroups are those listed in the lower row of the table,

$$\begin{array}{llllll} \mathcal{G}_o & SO(8) & SO(2n+2),\ n \geq 4 & SU(2n)\ n \geq 2 & SU(2n+1) & E_6 \\ \hat{\mathcal{G}}_0 & G_2 & SO(2n+1) & Sp(2n) & SO(2n+1) & F_4 \end{array} \tag{2.15}$$

Note that the groups in the upper and lower rows of (2.15) are simply-laced and non-simply-laced respectively, and that the only compact simple Lie groups not in the table are the simply-laced groups E_7 and E_8. The action of $\mathcal{T} = S_2$ on the representations is to change them into their complex conjugates for $SU(n)$ and E_6 and to interchange the two kinds of spinor representations for $SO(2n)$. The group W^0 may be identified in two ways. As the group of inner permutations of the roots, W^0 may be identified as the Weyl group of G_0, i.e. the group generated by the reflexions of the roots in the planes orthogonal to them and implemented by the group elements

$$W_0^\alpha = \exp i\frac{\pi}{2}(E^\alpha + E^{-\alpha}) \ , \tag{2.16}$$

and as the Cartan-preserving group of inner automorphisms, W^0 may be identified as $W^0 = N^0/C^0$, where N^0 is the normalizer in $\mathcal{G}_o$ of the Cartan algebra and C^0 is the centralizer. In other words, the Weyl group has two equivalent definitions, (2.16) and N^0/C^0.

3. PROPERTIES OF KM ALGEBRAS

The generalization[2,3] of simple compact Lie algebras G_0 to (untwisted affine) KM-algebras is

$$[T_m^a, T_n^b] = i f^{ab}{}_c T_N^c + m\delta^{ab}\delta_N K \ , \quad n, m \in \mathbf{Z},\ N = n + m, \tag{3.1}$$

where K is a central charge (commutes with all T_m^a) and $\delta_N = 0, 1$ for $n \neq 0$, $n = 0$. It is not actually necessary for n, m to be integers (elements of any additive group such as the half-integers or the real line would do for consistency) but the condition $m, n \in \mathbf{Z}$ reflects the compactness of the Fourier-transformed space. The Cartan form of (3.1) is

$$[H_m^i, H_n^j] = mK\delta^{ij}\delta_N, \quad [H_m^i, E_n^\alpha] = \alpha^i E_N^\alpha, \quad [E_m^\alpha, E_n^{-\alpha}] = \frac{2}{\alpha^2}(\alpha.H_N + mK\delta_N), \tag{3.2}$$

with further obvious relations for the E^α_m among themselves, and it is worth noting that even in the abelian case ($E^\alpha_m = 0$) the algebra (3.2) is non-trivial, and is in fact the algebra of the bosonic string in the light-cone gauge. It is usual to adjoin to any KM algebra a scale operator D, defined up to a central addition by

$$[D, T^a_m] = -mT^a_m \ . \tag{3.4}$$

Then because the H^i_m do not commute, and the H^i_0 and D determine the E^α_m uniquely, it is convenient to use

$$\{D, H^i_0, K\} \tag{3.5}$$

as the KM-Cartan subalgebra. Thus, apart from the trivial central term K, the KM-Cartan subalgebra contains only one more operator than the ordinary Cartan subalgebra.

There are some important Lie subalgebras of the KM-algebra, namely the Lie algebra

$$G_0 = \{H^i_0, E^\alpha_0\} \equiv \{H^i, E^\alpha\} \ , \tag{3.6}$$

on which the KM algebra is based, and the $SU(2)^\alpha_n$ subalgebras with generators

$$\{E^\alpha_n, E^{-\alpha}_{-n}, \frac{2}{\alpha^2}(\alpha.H_0 + nK)\} \ . \tag{3.7}$$

[The construction for $SU(2)^\alpha_n$ can actually be extended to algebras G_n isomorphic to G_0 by defining them as the closure under commutation of the generators E^β_n, $E^{-\beta}_{-n}$, $\frac{2}{\beta^2}(\beta.H_n + nK)$, where the β are the l primitive (simple) roots, but this extension will not be needed here].

The representations of the KM-algebras (3.1) in which we shall be interested are the so-called <u>highest-weight</u> (unitary) representations defined by self-adjoint conditions

$$(T^a_m)^\dagger = T^a_{-m} \ , \quad K^\dagger = K \geq 0 \tag{3.8}$$

and a vacuum $|0>$ for which

$$T^a_m|0>= 0, \ m > 0 \quad \text{and} \quad T^a_0|0>= U(T^a_0)|0> \ , \tag{3.9}$$

where $U(T^a_0)$ is a unitary representation of the Lie algebra G_0. (Thus the vacuum has the degeneracy of $U(T^a_0)$.) The unitarity of the representations of the $SU(2)^\alpha_n$

groups generated by the algebras in (3.7) implies that the central charge must be quantised according to

$$K = n\frac{\lambda^2}{2}\ , \quad \text{where} \quad n \in \mathbf{Z} \tag{3.10}$$

and λ is a long root. The multiple n is called the level of the representation. For non-trivial highest weight representations $n \geq 1$ and for the lowest non-trivial level, $K = K_{\min} = \frac{\lambda^2}{2}$, it can be shown that the permissible vacuum representations $U(T_0^a)$ of G_0 are just the fundamental representations i.e. those with highest weights $w^{(g)}$. Thus the irreducible KM representations fall into the same congruency classes as the Lie group representations. Finally it should be noted that the group elements

$$\exp 2\pi i D \quad \text{and} \quad \exp 2\pi i(\bar{w}.H) \tag{3.11}$$

of KM-algebras are central and are thus fixed for each irreducible representation.

4. THE GROUP OF CARTAN-PRESERVING AUTOMORPHISMS OF KM-ALGEBRAS

After these preliminaries we now proceed to the main purpose of the paper which is to consider the group $\tilde{N}$ of Cartan-preserving automorphisms and the Weyl group W of KM-algebras. Let us first consider the group $\tilde{N}$. Since the KM algebra is an invariant subalgebra of the combined D-KM system it is preserved by any automorphism and thus the action of $\tilde{N}$ must be of the form

$$H_0^i \to R^i_j(H_0^j + u^j K)\ , \quad D \to \lambda D + \vec{v}.\vec{H}_0 + \eta K\ , \quad K \to K\ , \tag{4.1}$$

where R^i_j, u^i, v^i, λ and η are parameters. Since the space $\{H_n\}$ is the unique eigenspace of H_0^i and D with eigenvalues zero and $\{n\}$ respectively, (with each n occuring once and only once), one sees from (4.1) that the set $\{H_n\}$ must transform into itself under $\tilde{N}$, and hence, by the preservation of the spectrum of D, the coefficient λ must be unity. Furthermore, for $\lambda = 1$, one sees from the KM commutators that the only permitted $\tilde{N}$-transformation of the H_n^i and E_n^α is (up to phases)

$$H_n^i \to R^i_j H_n^j\ , \ n \neq 0 \quad \text{and} \quad E_m^\alpha \to E^{\overline{\alpha}}_{m+(v.\overline{\alpha})} \quad \text{where} \quad \overline{\alpha}^i = (R^{-1})^i_j \alpha^j \tag{4.2}$$

and from the KM algebra relation

$$\begin{aligned}\frac{2}{\alpha^2}(\alpha.H_0 + mK) = [E_m^\alpha, E_{-m}^{-\alpha}] &\to [E^\alpha_{m+(v.\alpha)}, E^{-\alpha}_{-(m+(v.\alpha))}] \\ &= \frac{2}{\alpha^2}(\alpha.H_0 + (m + v.\alpha)K)\ ,\end{aligned} \tag{4.3}$$

one then sees that the vectors $\vec{u}$ and $\vec{v}$ in (4.1) must be identical. Thus, finally, the automorphisms $\tilde{N}$ of the Cartan subalgebra must take the form

$$H_0^i \to R_j^i(H_0^j + v^j K)\ , \quad D \to D + \vec{v}.\vec{H}_0 + \eta K\ . \tag{4.4}$$

In these transformations the matrices R_j^i are easily identified as the matrices of the Cartan-preserving automorphisms $\tilde{N}^0$ of the ordinary Lie algebra, and the structure of the group $\tilde{N}$ is easily seen to be a semi-direct product of the form

$$\tilde{N} = \tilde{N}^0 \wedge \tilde{\Lambda}\ , \tag{4.5}$$

where $\tilde{\Lambda}$ is the $(l+1)$-parameter invariant subgroup of transformations

$$H_0^i \to H_0^i + v^i K\ , \quad D \to D + \vec{v}.\vec{H}_0 + \eta K\ . \tag{4.6}$$

The group $\tilde{\Lambda}$ in turn is easily seen to be a semi-direct product, namely

$$\tilde{\Lambda} = \tilde{A} \wedge \tilde{T}\ , \tag{4.7}$$

where $\tilde{A}$ is the l-parameter subgroup

$$H_0^i \to H_0^i + v^i K\ , \quad D \to D + \vec{v}.\vec{H}_0 + \frac{v^2}{2}K \tag{4.8}$$

and $\tilde{T}$ is the one-parameter subgroup $D \to D + \eta K$. Hence, the full structure of $\tilde{N}$ is

$$\tilde{N} = \tilde{N}^0 \wedge (\tilde{A} \wedge \tilde{T})\ , \tag{4.9}$$

where $\tilde{N}^0$ is the group of automorphisms of the ordinary Lie algebra and $\tilde{A} \wedge \tilde{T}$ is a new group characteristic of the KM case. The group $\tilde{T}$ corresponds, of course, only to a trivial change of origin of D, so the essential KM feature is the existence of the group $\tilde{A}$. We now discuss the group $\tilde{A}$ in more detail.

Because of the discreteness of the root-spectrum, the parameters $\vec{v}$ (and η) must satisfy quantization conditions but since these depends on the Lie group in question and are absent when the spectrum of D is continuous, let us first identify the group $\tilde{A}$ for general $\vec{v}$. The identification is made by observing that the transformations (4.8) preserve the form

$$2KD - H_0^2\ , \tag{4.10}$$

and that this form is a Galilean one. This identifies $\tilde{A}$ as the group of Galilean accelerations in the l-dimensional Euclidean root space.

The fact that the metric of (4.10) is Galilean may be seen either by noting its analogy with the form $2mE - p^2$ of non-relativistic mechanics, or by noting that it is a Minkowski metric in an $(l+2)$-dimensional space with lightlike vectors D and K, one of which (K) is kept fixed. Indeed it is easy to see that the transformations (4.8) can be implemented by the unitary operators

$$\exp i\vec{v}.\vec{X}\ , \quad \text{where} \quad [X^i, H_0^j] = iK\delta^{ij}\ , \quad [X^i, D] = iH_0^i \tag{4.11}$$

and the algebra in (4.11) will be recognised as both the algebra of the position, momentum and energy in non-relativistic quantum mechanics, and the algebra of the centre-of-mass, total momentum and energy of the left or right-moving relativistic closed string in the light-cone gauge[4)] (see section 8).

For (untwisted) KM-algebras (and for the closed string), however, the spectrum of D is integral, and hence $\vec{v}$ is not free but quantised. In fact, from (4.2), one sees that for the E_m^α to transform into themselves it is necessary and sufficient that

$$(v.\alpha) \in \mathbf{Z}, \quad \text{or equivalently} \quad \vec{v} \in \Gamma_{\tilde{w}}\ . \tag{4.12}$$

So the quantization condition for $\vec{v}$ is that it be an element of the coweight lattice. Thus, finally, the group $\tilde{N}$ of Cartan-preserving automorphisms is given by (4.9), where $\tilde{A}$ is given by (4.8) with $\vec{v} \in \Gamma_{\tilde{w}}$.

To see the effect of these automorphisms on the congruency classes of representations one considers the lowest-level representations $K = \frac{\lambda^2}{2}$ for which

$$\vec{H}_0 \rightarrow \vec{H}_0 + \vec{v}\frac{\lambda^2}{2}\ , \quad \vec{v} \in \Gamma_{\tilde{w}}\ . \tag{4.13}$$

Since from equations (2.9) one has the inclusion $\lambda^2\Gamma_{\tilde{w}} \subseteq 2\Gamma_w$ with equality if, and only if, the group is simply-laced, one sees that for the simply-laced groups the automorphisms connect all the congruency classes (e.g. all the $l+1$-fundamental representations of $SU(n)$). But for the non-simply-laced groups $SO(2n+1)$ and $Sp(2n)$ they connect only congruent representations and for the non-simply-laced groups with trivial centre (G_2 and F_4) they only connect representations which are congruent modulo the long roots. Note that the $\tilde{A}$ automorphisms also permute the groups $SU(2)^{\alpha}_{n}$ and change G_0 into the group $\{E^{\alpha}_{(\alpha.v)}, E^{-\alpha}_{-(\alpha.v)}, \ \alpha.(H_0+\vec{v}K)\}$.

Let us finally consider the action of the automorphisms $\tilde{A}$ on the vectors in weight-space, which is defined as the $(l+2)$-dimensional space spanned by eigenvalues $P = (d, \vec{p}, k)$ of the Cartan operators $\{D, \vec{H}_0, K\}$, and which, on account of (4.10) has a Minkowski-metric with inner products of the form

$$P.P' = \vec{p}.\vec{p}' - (dk' + d'k) \ . \tag{4.14}$$

The ordinary permutation group $\tilde{N}^0$ of roots α, corresponds of course to rotations of $\vec{p}$ with d, k fixed so what we are really interested in are the transformations of P induced by $\tilde{A}$ in (4.8). It is easy to see that these are of the form

$$(d, \vec{p}, k) \rightarrow (d + \vec{v}.\vec{p} + \frac{v^2}{2}k, \vec{p} + \vec{v}k, k) \tag{4.15}$$

and that they can be written in the more compact form

$$P \rightarrow P - \frac{2(P.V)}{V^2}V \ , \quad \text{where} \quad V = (1, \vec{v}, 0). \tag{4.16}$$

From (4.16) one sees that they are the exact analogues in Minkowski space of reflexions in Euclidean space. In particular they preserve the norm ($P'^2 = P^2$) and thus are Lorentz transformations. However, they also preserve the light-like vectors $(0, 0, k)$ so they belong to 'the little group of a light-like vector' which is known to be a Galilean subgroup of the Lorentz group.

5. INNER AND WEYL GROUPS OF CARTAN-PRESERVING AUTOMORPHISMS

So far we have considered the group $\tilde{N}$ of general Cartan-preserving automorphisms of KM algebras which, modulo the Cartan preserving automorphisms $\tilde{N}^0$ of ordinary Lie algebras and the trivial translations $D \rightarrow D + \eta K$, consisted of the Galilean transformations $\tilde{A}$ in (4.8) with $\vec{v} \in \Gamma_{\tilde{w}}$. Let us now consider the case in which the automorphisms $\tilde{A}$ are inner.

Since the integer spacing of D and the root spacing of $\vec{H}$ is preserved by KM algebras, it is clear that necessary conditions for the automorphisms (4.8) to be inner are that

$$\Delta\vec{H}_0 \equiv \vec{v}K \in \Gamma_{\alpha} \quad \text{and} \quad \Delta D \equiv v.H_0 + \frac{v^2}{2}K \in \mathbf{Z} \ . \tag{5.1}$$

But since the trivial representation $\vec{H}_0 = 0$ of G_0 is one of the permitted representations for the lowest level $K = K_{\text{min}} = \frac{\lambda^2}{2}$, and since $K = nK_{\text{min}}$, where $n \in \mathbf{Z}$, the second condition in (5.1) splits into two separate conditions

$$v.H_0 \in \mathbf{Z} \quad \text{and} \quad \frac{v^2}{2}K \in \mathbf{Z}\,, \tag{5.2}$$

and, since in general H_0 can be any weight, these conditions and the first condition in (5.1) may be written as

$$\vec{v}K \in \Gamma_\alpha, \quad \vec{v} \in \Gamma_{\tilde{\alpha}} \quad \text{and} \quad \frac{v^2}{2}K \in \mathbf{Z}\,. \tag{5.3}$$

We now show that the second condition in (5.3) implies the other two and thus is the only condition that is really necessary. First, we show that, irrespective of the value of K, the first two conditions in (5.3) imply the third. Indeed the result follows from the observation that

$$\vec{v} = K^{-1}\sum n_i\alpha_i = \sum m_i\tilde{\alpha}_i \quad \text{where} \quad m_i, n_i \in \mathbf{Z}\,, \tag{5.4}$$

implies that

$$v^2K^2 = \sum_i n_i^2(\alpha_i.\alpha_i) + 2\sum_{i>j} n_in_j(\alpha_i.\alpha_j) = 2K\{\sum_i n_im_i + \sum_{i>j} n_im_j(\alpha_i.\tilde{\alpha}_j)\} \tag{5.5}$$

where all the quantities in the curly bracket are integers. Second, we show that, since $K_{\text{min}} = \frac{\lambda^2}{2}$, the middle condition in (5.3) implies the first. Indeed from equations (2.9) we have $\Gamma_{\tilde{\alpha}} \subseteq \frac{2}{\lambda^2}\Gamma_\alpha$ and hence

$$\begin{aligned} \vec{v} \in \Gamma_{\tilde{\alpha}} &\Rightarrow \vec{v} \in \frac{2}{\lambda^2}\Gamma_\alpha = K_{\text{min}}^{-1}\Gamma_\alpha \\ &\Rightarrow \vec{v}K \in \Gamma_\alpha\,. \end{aligned} \tag{5.6}$$

It is worth noting that the condition $\frac{v^2}{2}K \in \mathbf{Z}$ in (5.3) is a consequence of the other two for any value of K, whereas the first condition in (5.3) is a consequence of the second only because $K_{\text{min}} = \lambda^2/2$. In any case, for untwisted KM algebras the final result is that the necessary condition (5.1) for the automorphisms to be inner reduces to

$$\vec{v} \in \Gamma_{\tilde{\alpha}}\,. \tag{5.7}$$

We shall now show that (5.7) is also a sufficient condition by constructing the operators in the KM group that implement the transformations with parameters $\vec{v} = \tilde{\alpha}$. For this we consider the elements

$$W_n^\alpha(\theta) = \exp \frac{i\theta}{2}(E_n^\alpha + E_{-n}^{-\alpha}) \tag{5.8}$$

of the Lie subgroups $SU(2)_n^\alpha$ of (3.7). From the KM algebra one easily verifies that

$$\begin{aligned}(W_n^\alpha(\theta))^\dagger H_0^i\, W_n^\alpha(\theta) = H_0^i + \alpha^i[\frac{i}{2}(E_n^\alpha - E_{-n}^{-\alpha})]\sin\theta \\ - \frac{\alpha^i}{\alpha^2}(\alpha.H_0 + nK)(1-\cos\theta)\end{aligned} \tag{5.9}$$

$$\begin{aligned}(W_n^\alpha(\theta))^\dagger D\, W_n^\alpha(\theta) = D - n[\frac{i}{2}(E_n^\alpha - E_{-n}^{-\alpha})]\sin\theta \\ + \frac{n}{\alpha^2}(\alpha.H_0 + nK)(1-\cos\theta)\end{aligned} \tag{5.10}$$

and hence for $\theta = \pi$ (when the $W_n^\alpha(\theta)$ become the elements of the Weyl group of $SU(2)_n^\alpha$) one has

$$\begin{aligned}(W_n^\alpha(\pi))^\dagger H_0^i\, W_n^\alpha(\pi) &= H_0^i - \tilde{\alpha}^i(\alpha.H_0 + nK) \\ (W_n^\alpha(\pi))^\dagger D\, W_n^\alpha(\pi) &= D + n\frac{\tilde{\alpha}^2}{2}(\alpha.H_0 + nK)\ .\end{aligned} \tag{5.11}$$

Thus the transformations $W_n^\alpha(\pi)$ preserve the Cartan subalgebra $\{D, \vec{H}_0, K\}$ and in fact are just automorphisms of the form (4.9) which are mixtures of ordinary Weyl rotations induced by α and Galilean accelerations $\tilde{A}$ induced by $n\tilde{\alpha}$. Hence if one cancels the ordinary Weyl rotations by defining

$$U_n^\alpha = W_n^\alpha(\pi)(W_0^\alpha(\pi))^{-1} \tag{5.12}$$

one finds that

$$(U_n^\alpha)^{-1} H_0^i\, U_n^\alpha = H_0^i - n\tilde{\alpha}^i K, \qquad (U_n^\alpha)^{-1} D\, U_n^\alpha = D - n\tilde{\alpha}.H_0 + \frac{(n\tilde{\alpha})^2}{2}K \tag{5.13}$$

and for $n = 1$ these are just the generators of the Galilean transformations (4.8) with $\vec{v} \in \Gamma_{\tilde{\alpha}}$ as required. Thus, finally, $\vec{v} \in \Gamma_{\tilde{\alpha}}$ is the necessary and sufficient condition for the Cartan-preserving automorphisms to be inner.

Note that for $\vec{v} \in \Gamma_{\tilde{\alpha}}$ the 'reflexions' (4.15) of the previous section reduce exactly to the 'reflexions' by which previous authors[3)] defined the Weyl group W of

KM algebras. Thus we may call the group of Cartan-preserving inner automorphisms the Weyl group W, and, just as in the case of ordinary Lie algebras, it may be regarded either as the group generated by the reflexions (4.15) (4.16) with $\vec{v}$ in $\Gamma_{\tilde{\alpha}}$ or, from (5.13), as the group $W = N/C$ where N and C are the normalizer and centralizer (in the group of inner automorphisms) of the Cartan subalgebra $\{D, \vec{H}_0, K\}$ respectively. Note also that the inner automorphism relation

$$\exp(i\tilde{\alpha}.X) = \exp(\frac{i\pi}{2}(E_1^{\alpha} + E_{-1}^{-\alpha}))\exp(\frac{-i\pi}{2}(E_0^{\alpha} + E_0^{-\alpha})) \tag{5.14}$$

could be regarded as a formal definition of the operator $\vec{X}$. From this point of view $\vec{X}$ is then defined by the KM algebra itself, but since it is independent of α, it acquires a universality that transcends the particular KM algebra used to define it. Indeed, as we have seen, $\vec{X}$ can be used to formally generate the outer automorphisms also, by extending the range of the parameter $\vec{v}$ in $\exp(i\vec{v}.X)$ from $\Gamma_{\tilde{\alpha}}$ to $\Gamma_{\tilde{w}}$. Furthermore, if the spectrum of D is taken to be continuous the range of $\vec{v}$ can be extended to the continuum. In particular, $\exp(i\vec{v}.\vec{H}_0)$ for continuous $\vec{v}$ generates automorphisms of the abelian KM (string) algebra. In this sense $\vec{X}$ may be regarded as a remnant of the non-abelian part of the KM algebra in the limit when the strictly non-abelian part E_n^{α} is contracted to zero.

6. EXTENSION TO THE VIRASORO ALGEBRA

It is well-known that every KM-algebra admits a Virasoro automorphism and that it is implemented by generators L_n which satisfy the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m} , \tag{6.1}$$

where c is an arbitrary constant, and have the action

$$[L_n, T_m^a] = -mT_{m+n}^a , \tag{6.2}$$

on the KM-algebra, and an interesting feature of the Cartan-preserving automorphisms $\tilde{N}$ (both inner and outer) is that they can be extended to include the Virasoro algebra. In fact, it is easy to verify that, if v denotes the usual parameter of Galilean accelerations, the required extension of $\tilde{N}$ to the combined KM and V system is

$$L_n \to L_n + \vec{v}.H_n + \frac{v^2}{2}K\delta_n , \tag{6.3}$$

and that it is unique. One sees from (6.3) that L_0 has the same transformation law as D in (4.8), but this is not surprising since L_0 and D coincide in the sense that $L_0 - D$ is central to the whole KM-V system.

It is worth noting that the automorphism (6.3) is not an automorphism of the Virasoro algebra L_n alone, but transforms the Virasoro algebra into a mixture of itself and the roots $\vec{H}_n, K$ of the KM-algebra. In fact the spaces spanned by $\{L_n, \vec{H}_n, 0\}$ for fixed $n \neq 0$ and by $\{L_0, H_0, K\}$ form reducible, but not fully reducible, representations of the Galilean accelerations (4.8) and indeed of the full group $\tilde{N} = \tilde{N}^0 \wedge (\tilde{A} \wedge \tilde{T})$. Thus with respect to $\tilde{N}$ any combined KM-V-algebra decomposes into the representations $\{L_0, \vec{H}_0, K\}$ the representations $\{L_n, \vec{H}_n, 0\}$ for fixed $n \neq 0$, and the representation $\{E_n^\alpha\}$ for all n, α. Further, apart from the usual decomposition of the Cartan algebras according to short and long roots, which yields two $\tilde{N}^0$ orbits, the representations each consist of a single orbit. Thus, finally, the action of $\tilde{N}$ on the KM-V-algebra is analogous to the action of $\tilde{N}^0$ on an ordinary Lie algebra, indeed just extends the action of $\tilde{N}^0$ on G_0 by extending the $\tilde{N}^0$-orbits to include (all) the different values of n.

For highest weight representations of KM-algebras it is well-known that the V-generators L_n can be realized as bilinears in the KM-generators, so that the Virasoro automorphism can be thought of as an internal automorphism. The realization is through the so-called Sommerfield-Sugawara (SS) construction[3,6,8]

$$\begin{aligned}(K + \frac{1}{2}Q)L_n^{SS} &= \sum_{pq} T_p^a T_q^a \theta(q-p)\delta(p+q-n) \\ &= \sum_{p,q} (\vec{H}_p.\vec{H}_q + \sum_\alpha \frac{\alpha^2}{2} E_p^{-\alpha} E_q^\alpha)\theta(q-p)\delta(p+q-n)\end{aligned} \tag{6.4}$$

where Q is the group-invariant constant defined in equation (2.2) and $\theta(s)$, the normal-ordering step-function, is defined as $\theta(s) = 0, \frac{1}{2}, 1$ for $s > 0, = 0, < 0$ respectively, and it is instructive to verify that for the L_n^{SS} the transformations (6.3) are just those induced by the $\tilde{N}$-automorphisms of the KM algebra. Actually, since the ordinary Weyl rotations are trivial for (6.3) and since for the Galilean accelerations v, the change in the H-term in (6.4) is

$$\Delta(\sum_{p,q} \vec{H}_p.\vec{H}_q \theta\delta) = \Delta(\frac{1}{2}\vec{H}_0^2\delta_n + \vec{H}_0.\vec{H}_n(1-\delta_n)) = K(\vec{v}.\vec{H}_n + \frac{v^2}{2}K\delta_n) \tag{6.5}$$

the only non-trivial part of the verification is to show that the change in the E-term in (6.4) due to $\vec{v}$ is

$$\Delta(\sum \frac{\alpha^2}{2} E_p^{-\alpha} E_q^{\alpha} \theta\delta) = \frac{1}{2} Q(\vec{v}.\vec{H}_n + \frac{v^2}{2} K\delta_n). \tag{6.6}$$

To show this one notes that the change $E_q^\alpha \to E_{q+m}^\alpha$, where $m = (\alpha.v)$, induced in the KM generators E by the $\vec{v}$-transformations may be absorbed in the θ-function so that

$$\Delta(\sum_{\alpha,p,q} \alpha^2 E_p^{-\alpha} E_q^{\alpha} \theta(q-p)\delta(q+p-n)) = -\sum_{\alpha,p,q} \alpha^2 E_p^{-\alpha} E_q^{\alpha} \chi(q-p,m)\delta(q+p-n) \tag{6.7}$$

where $\chi(q-p,m)$ is the difference

$$\chi(q-p,m) = \theta(q-p) - \theta(q-p-2m). \tag{6.8}$$

But since the factor $\chi\delta$ makes the sum over p,q finite, and $\chi\delta$ is odd in $q-p$, one may make, without change in (6.5), the replacement

$$E_p^{-\alpha} E_q^{\alpha} \to \frac{1}{2}[E_p^{-\alpha}, E_q^{\alpha}] = \frac{1}{\alpha^2}(-\alpha.H_n - qK\delta_n) \tag{6.9}$$

and then (6.7) reduces to

$$\Delta(\sum \alpha^2 E^{-\alpha} E^{\alpha} \theta\delta) = \sum_{\alpha} [(\alpha.H_n)x + K\delta_n y] \tag{6.10}$$

where x and y are the numerical factors

$$x = \sum_q \chi(2q-n,m) = m \qquad y = \sum_q q\chi(2q,m) = \frac{m^2}{2}. \tag{6.11}$$

Then recalling that m is actually $(\alpha.v)$ and using the completeness relation (2.4) for the α^i's one has

$$\begin{aligned}\Delta(\sum \alpha^2 E^{-\alpha} E^{\alpha} \theta\delta) &= \sum_{\alpha} [(v.\alpha)(\alpha.H_n) + \frac{(v.\alpha)(\alpha.v)}{2} K\delta_n] \\ &= Q(v.H_n + \frac{v^2}{2} K\delta_n)\ ,\end{aligned} \tag{6.12}$$

as required.

A corollary of the verification for L_n^{SS} is that for any L_n satisfying (6.1) (6.2), the quantities

$$KL_n - \vec{H}_0.\vec{H}_n, n \neq 0\,, \quad 2KL_0 - H_0^2\,,$$
$$QL_n - \sum(\frac{\alpha^2}{2}E_p^{-\alpha}E_q^{\alpha}\theta\delta) \quad \text{and} \quad \vec{H}_p.\vec{H}_q,\ p,q \neq 0\,, \tag{6.13}$$

are separately invariant with respect to the Cartan-preserving group $\tilde{N}$, whereas only the sum of them is invariant with respect to the whole KM-algebra. This observation explains why the combination $2KD - H_0^2$ introduced in (4.10) is the invariant of the Cartan algebra with respect to the group $\tilde{N}$, whereas

$$(2K+Q)D - \vec{H}_0^2 - 2\sum_{n=1}^{\infty}\vec{H}_{-n}.\vec{H}_n - \sum_n\sum_\alpha \alpha^2 E_{-n}^{-\alpha}E_n^{\alpha}\theta(n) \tag{6.14}$$

with a different ratio of coefficients for D and $\vec{H}_0^2$ is the invariant with respect to the full KM-algebra. The interaction with the non-abelian part renormalizes D and $\vec{H}_0$ in a different manner!

7. USE OF THE NORMALIZER IN THE VERTEX CONSTRUCTION

The operator $\vec{X}$ is also used in the standard vertex construction[3)] of the non-abelian part of the KM-algebras, but it is interesting to note that in fact it is not $\vec{X}$ itself but only the exponentiated quantities $U_1^{\beta} \in \tilde{A}$ in equation (5.8) which are necessary for the vertex construction. For example, for the simply-laced untwisted KM-algebras the vertices may be written as the products

$$V_\beta(z) = \gamma_\beta \overset{0}{V_\beta}(z) \prod_{n\geq 1} V_n^{\beta}\,, \tag{7.1}$$

where the factors commute and are defined as follows: The γ_β are constant (z-independent) elements of a Clifford algebra

$$\gamma_\alpha\gamma_\beta = \epsilon(\alpha,\beta)\gamma_{\alpha+\beta} \quad \alpha \neq \beta\,, \quad \gamma_\alpha^{\dagger} = \gamma_{-\alpha} = \gamma_\alpha^{-1}\,, \tag{7.2}$$

defined on the root diagram (but not necessarily extended to the root lattice), the structure constants $\epsilon(\alpha,\beta)$ being those for the non-abelian part of G_0 i.e.

$$\epsilon(\alpha,\beta) = \mathrm{tr}((\mathrm{E}_0^{\alpha+\beta}[\mathrm{E}_0^{-\alpha}, \mathrm{E}_0^{-\beta}]) \,. \tag{7.3}$$

The $V_n^\beta(z)$ are products of non-commuting operators i.e.

$$V_n^\beta(z) = \mathcal{V}_{-n}^\beta(z)\mathcal{V}_n^\beta(z) , \tag{7.4}$$

where $\mathcal{V}_n^\beta = \exp(iz^n(\beta.H_n))$ satisfy the Heisenberg-like commutation relations

$$\mathcal{V}_n^\beta(z)\mathcal{V}_{-n}^\alpha(\eta) = \mathcal{V}_{-n}^\alpha(\eta)\mathcal{V}_n^\beta(z)\exp[i(\alpha.\beta)(z/\eta)^n] . \tag{7.5}$$

Finally, the $\overset{0}{V}_\beta(z)$, which are the factors in which we are interested, are

$$\overset{0}{V}_\beta(z) = U_1^\beta(C^\beta)^{\ln z} , \tag{7.6}$$

where $C^\beta = \exp(i\beta.H_0)$ and U_1^β is defined in equation (5.12). Thus the $\overset{0}{V}_\alpha(z)$ satisfy the analogous Heisenberg-like relations

$$\overset{0}{V}_\alpha(z)\overset{0}{V}_\beta(\eta) = \overset{0}{V}_\alpha(\eta)\overset{0}{V}_\beta(z)\left[z/\eta\right]^{\alpha.\beta} \tag{7.7}$$

and the full vertex $V_\alpha(z)$ satisfies

$$V_\alpha(z)V_\beta(\eta) = V_\beta(\eta)V_\alpha(z)\left[z-\eta\right]^{\alpha.\beta} . \tag{7.8}$$

Equations (7.8) are the commutation relations of the Fourier-transformed KM-algebra (3.2).

The points we wish to emphasize here, however, are the following. First, since the U_1^β and C^β are elements of the acceleration and Cartan subgroups A and C of N respectively, one sees that the essential part $\overset{0}{V}_\beta(z)$ of the vertex is an element of N. Second, since the rest of the vertex depends only on the γ_α and the $\vec{H}_n$, one sees that the total vertex is constructed from three ingredients, namely the Lie algebra G_0, the abelian KM (string) algebra $\{\vec{H}_n, K\}$ and the KM Weyl group W.

8. SPACE-TIME SIGNIFICANCE OF THE NORMALIZER FOR ABELIAN KM ALGEBRAS

We now show that the Cartan-preserving group $\tilde{N}$ and its Weyl subgroup W for KM algebras have a remarkable space-time significance within the context of string theory. For this we recall[4] that the transverse part of the string algebra in the light-cone gauge is just an abelian KM-algebra,

$$[H_m^i, H_n^j] = m\delta^{ij}\delta_N K , \quad [D, H_m^i] = -mH_m^i , \; i,j = 1,...,l , \tag{8.1}$$

to which we can adjoin the $\tilde{N}$ group generator X^i of (4.11) defined as

$$[X^i, H_0^j] = iK\delta^{ij} \qquad [X^i, H_n^j] = 0, \; n \neq 0 \; , \tag{8.2}$$

and we first notice that the generators M^{ij} of orthogonal transformations of the vectors X^i, H_n^i can be constructed from these operators themselves by the standard formula,

$$M^{ij} = X^{[i}H_0^{j]} + \sum_{n\neq 0} \frac{1}{n} H_{-n}^{[i} H_n^{j]} \; , \quad i,j = 1,...,l \; , \tag{8.3}$$

where $[i\;j]$ denotes anti-symmetrization. Now in the light-cone gauge formulation of string theory, the l-dimensional weight space $\mathbf{R}_l$ is regarded as a transverse Euclidean subspace of an $(l+2)$-dimensional Minkowski space, whose remaining two dimensions are spanned by two light-like vectors (k^μ and κ^μ say) orthogonal to $\mathbf{R}_l$, and the oscillator KM algebra (8.1) is extended to the two extra dimensions by defining

$$k.H_n = \delta_n K \quad \text{and} \quad \kappa.H_n = L_n^{SS}(\vec{H})/2K, \qquad k.X = 0 \quad \text{and} \quad [\kappa.X, K] = i \, , \tag{8.4}$$

where $L_n^{SS}(\vec{H})$ denotes the SS-Virasoro operators of the transverse string algebra (8.1). When the Minkowski space has the critical dimension $l+2=26$ the quantities

$$M^{\mu\nu} = X^{[\mu}P^{\nu]} + \sum_{n\neq 0} \frac{1}{n} H_{-n}^{[\mu} H_n^{\nu]} \quad \text{and} \quad P^\mu = H_0^\mu, \; \mu = 0,...,l+1 \tag{8.5}$$

close to form a Poincaré algebra and are identified as the generators of the physical Poincaré group.

The interesting point is that the little group of this physical Poincaré group that leaves the scalar $k.P$ invariant is easily seen to be generated by P^μ and $M^{\mu\nu}k^\nu$, and if one writes these generators in terms of the string algebra one finds that they are

$$\kappa.P = L_0 = D, \; P^i = H_0^i, \; k.P = K, \; M^{ij} \quad \text{and} \quad M^{i+} = X^i \; . \tag{8.6}$$

Thus they are exactly the generators of the Cartan-preserving group $\tilde{N}$ (which exists, incidentally, whether or not $l+2=26$). From this we see that, for the string, the Cartan-preserving group $\tilde{N}$ is the little group of a light-like vector in Minkowski space. Thus $\tilde{N}$ plays the dual role of an internal and a space-tme symmetry group!

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