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A Hamiltonian Formalism For Bosonic Membranes

B.P.Dolan

Department of Mathematical Physics
St. Patrick's College
Maynooth, Ireland

and

School of Theoretical Physics
Dublin Institute for Advanced Studies
10 Burlington Road, Dublin 4, Ireland

ABSTRACT

An action for Bosonic membranes, which has no cosmological constant, is studied. The Hamiltonian formalism is developed, with a view to quantisation, using Dirac's method for constrained systems. The commutators of the independent canonical variables are evaluated in a co-ordinate gauge, at least to lowest order in $\hbar$.

Introduction

There has recently been much interest in membranes as a generalisation of strings in the extended object approach to fundamental physics [1-3]. The purpose of this paper is to explore the properties of an action for bosonic membranes, first proposed in [4]. This action differs from the more usual actions used for membranes in that it is polynomial in the dynamical fields (quartic plus quadratic) and does not require a cosmological constant. It is based on a generalisation of a conformally invariant σ model action in four dimensions developed in [5].

To quantise this membrane action, the Hamiltonian formalism is adopted and constraints are handled using Dirac's formalism. The Poisson algebra of the constraints is evaluated and first and second class constraints are identified. The first class constraints are transformed into second class constraints by choosing a gauge which eliminates all the longitudinal degrees of freedom, leaving only transverse co-ordinates as free variables. The Dirac brackets for the independent canonical variables are evaluated, at least to lowest order in $\hbar$, and are shown to be the same as the Poisson brackets, thus providing a convenient starting point for the quantum theory.

The Embedding

A bosonic membrane embedded in a D dimensional space-time,$(\mathcal{M}, g)$, with a D dimensional metric, g, sweeps out a three dimensional world volume,Σ, (two space and one time dimension). x^μ will denote co-ordinates in D dimensions $(\mu, \nu, \ldots = 0, 1, \ldots, D-1)$ and $g_{\mu\nu}$ the metric components. Co-ordinates on the world volume of the membrane will be denoted by σ^a $(a, b, \ldots = 0, 1, 2)$. The embedding, $x^\mu(\sigma)$, of the world volume of the membrane into space-time induces a metric on Σ,

$$G_{ab}(\sigma) = \partial_a x^\mu(\sigma) \partial_b x^\nu(\sigma) g_{\mu\nu}(x(\sigma)) \tag{1}$$

and a three dimensional connection (everything is assumed torsion free)

$$\begin{aligned} \gamma^a_{bc} &= \frac{1}{2} G^{ad}(G_{db,c} + G_{dc,b} - G_{bc,d}) \\ &= G^{ad} \partial_d x^\rho (\partial_b \partial_c x^\mu + \partial_b x^\nu \partial_c x^\tau \Gamma^\mu_{\nu\tau}) g_{\rho\mu} \end{aligned} \tag{2}$$

where $\Gamma^\mu_{\nu\rho}$ is the D dimensional Christoffel connection

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\tau}(g_{\tau\nu,\rho} + g_{\tau\rho,\nu} - g_{\nu\rho,\tau})$$

and G^{ab} is the inverse of G_{ab}. (There is a slight abuse of notation here, since $\Gamma^\mu_{\nu\rho}$ means $\Gamma^\mu_{\nu\rho}(x(\sigma))$ i.e. the Christoffel connection on $(\mathcal{M}, g)$ restricted to Σ which is, of course, distinct from γ^a_{bc}.)

Co-variant differentiation of tensors, T^μ, on $\mathcal{M}$ will be denoted by ; thus

$$T^\mu_{;\nu} = T^\mu{}_{,\nu} + \Gamma^\mu_{\nu\rho} T^\rho$$

while co-variant differentiation of tensors on Σ will be denoted by $\|$ thus

$$T^a_{\|b} = T^a{}_{,b} + \gamma^a_{bc} T^c.$$

A useful object in the sequel will be the projection operator

$$P^\mu_\tau(x(\sigma)) = G^{ab} \partial_a x^\nu \partial_b x^\mu g_{\nu\tau} \tag{3}$$

which has the property that

$$P^\mu_\tau P^\tau_\nu = P^\mu_\nu.$$

It is not difficult to prove the following identity for any smooth embedding

$$P^\mu_\nu \Box\, x^\nu = -P^\mu_\nu P^{\rho\tau} \Gamma^\nu_{\rho\tau} \tag{4}$$

where

$$\Box\, x^\mu = G^{ab}(\partial_a x^\mu)_{\|b}.$$

The Action

To determine the dynamics we introduce an independent metric on Σ, $H_{ab}(\sigma)$, which is a priori unrelated to $G_{ab}(\sigma)$. $H_{ab}(\sigma)$ will later be identified with $G_{ab}(\sigma)$ only through the dynamics. Denote the inverse of H_{ab} by H^{ab}.

The action is taken to be

$$L = \int_{\Sigma} d^3\sigma \mathcal{L}(\sigma)$$

with

$$\mathcal{L}(\sigma) = -\sqrt{-detH}\left(H^{ab}G_{ab} + \frac{1}{2}H^{ab}H^{cd}(G_{ab}G_{cd} - G_{ad}G_{bc})\right) \tag{5}$$

where the integral is over the world volume of the membrane. The dynamical variables are $H^{ab}(\sigma)$ and $x^{\mu}(\sigma)$, the latter appearing in the definition of $G^{ab}(\sigma)$ (equation (1)).

It was reported in [4] that this action is classically equivalent to the Nambu-Goto action ([6],[7]),

$$L = -6\int_{\Sigma} d^3\sigma\sqrt{-detG} \tag{6}$$

with the dynamical relation

$$G^{ab} = H^{ab} \tag{7}$$

A proof is given in the next section.

The introduction of the independent metric,H_{ab}, is similar in spirit to the course followed in [8] for the bosonic string Lagrangian. The Lagrangian (5) is one of a class of Lagrangians introduced in [4] with a view to avoiding the introduction of a cosmological constant into the dynamics.

Lagrangian Formalism

To determine the dynamics of (5), first vary H^{ab} in L to give

$$\frac{\delta L}{\delta H^{ab}} = \frac{1}{2}\sqrt{-detH}\,H_{ab}\big(H^{cd}G_{cd} + \frac{1}{2}H^{fg}H^{cd}(G_{fg}G_{cd} - G_{fc}G_{gd})\big)$$
$$- \sqrt{-detH}\big(G_{ab} + H^{cd}(G_{ab}G_{cd} - G_{ac}G_{bd})\big)$$

$$= 0.$$

Define a matrix $A^a_b = H^{ac}G_{cb}$, then this is equivalent to

$$A^2 - (1 + trA)A + \frac{1}{2}\Big(trA + \frac{1}{2}\big((trA)^2 - tr(A^2)\big)\Big)\mathbf{1}_{3\times3} = 0. \tag{8}$$

To solve this equation for A, note that any square matrix can be put into Jordan normal form by a similarity transformation [9]. Thus A can have one of three forms,

$$A = S^{-1}\begin{pmatrix} a_1 & 0 & 0 \\ 1 & a_1 & 0 \\ 0 & 1 & a_1 \end{pmatrix} S \qquad A = S^{-1}\begin{pmatrix} a_1 & 0 & 0 \\ 1 & a_1 & 0 \\ 0 & 0 & a_2 \end{pmatrix} S \qquad A = S^{-1}\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} S$$

It is not difficult to show that the first two forms cannot satisfy (8), hence only the last form is a possible solution. Since A satisfies a quadratic equation, there must exist numbers λ_1 and λ_2 (possibly complex) such that

$$(A - \lambda_1\mathbf{1})(A - \lambda_2\mathbf{1}) = 0.$$

Thus there are only two possible values for a_1, a_2 and a_3. By a suitable relabelling, the only possible combinations are; either $a_1 = a_2 = a_3 = \lambda_1$, or $a_1 = a_2 = \lambda_1$ and $a_3 = \lambda_2$. In the first case, equation (8) implies $a_1 = a_2 = a_3 = \lambda_1 = 1$ and $A = \mathbf{1}$, while in the second case $a_1 = a_2 = \lambda_1 = -1$ and $a_3 = \lambda_2 = 1$. We reject the second solution since it admits the possibility of H^{ab} having a different signature from G^{ab}.

Hence the only physically acceptable solution is $A = \mathbf{1}$, which gives

$$H^{ab} = G^{ab}.$$

Putting this back into (5) reproduces the Nambu-Goto action (6). Varying x^μ now leads to the Lagrangian equations of motion

$$\Box\, x^\mu = -P^{\nu\rho}\Gamma^\mu_{\nu\rho} \tag{9}$$

with $P^{\mu\nu}$ given in terms of x^μ, by (3) and (1). This is, of course, a dynamical equation and as such is more restrictive than the identity (4), though it is obviously compatable with (4). Essentially, the identity (4) says that the dynamics is all in the tranverse part of equation (9).

Hamiltonian Formalism

The first step towards quantisation of this system is to construct the Hamiltonian, using $\sigma = \tau$ as the time variable. Rather than using H^{ab} as dynamical variables, it is convenient (as is usual with co-ordinate independent systems [10]) to use lapse and shift functions, N and N^i, together with the two dimensional space-like metric h^{ij} , $(i,j = 1,2)$, defined by

$$h_{ij} = H_{ij} \qquad H^{ij} = h^{ij} - N^i N^j/N^2 \qquad h_{ij}N^j = H_{0i} \qquad N^i/N^2 = H^{0i}$$
$$H_{00} = -N^2 + h_{ij}N^i N^j \qquad H^{00} = -1/N^2 \qquad \sqrt{-detH} = N\sqrt{deth} \equiv N\sqrt{h}$$

The canonical momenta conjugate to x^μ, N, N^i and h^{ij} are

$$\pi_\mu = \frac{\delta L}{\delta \dot{x}^\mu} \qquad K = \frac{\delta L}{\delta \dot{N}} \qquad K_i = \frac{\delta L}{\delta \dot{N}^i} \qquad K_{ij} = \frac{\delta L}{\delta \dot{h}^{ij}}.$$

Immediately there are six primary constraints ([10],[11])

$$K \approx 0 \qquad K_i \approx 0 \qquad K_{ij} \approx 0 \tag{10}$$

($\approx$ means weakly zero, as defined in [11]).

Thus, the momenta conjugate to N, N^i and h^{ij} vanish and we find (suppressing repeated space-time indices) that the canonical Hamiltonian is

$$H_c(\tau) = \int_{\Upsilon(\tau)} d^2\sigma \left(\frac{N}{2\sqrt{h}}(\pi M^{-1}\pi) + \frac{N\sqrt{h}}{4}(\partial_i x M \partial_j x)h^{ij} + N^i(\pi\partial_i x) \right) \tag{11}$$

where

$$M_{\mu\nu} = 2(G_{ij}g_{\mu\nu} - \partial_i x^\rho \partial_j x^\tau g_{\rho\mu}g_{\tau\nu})h^{ij} + 2g_{\mu\nu}$$

with G_{ij} given by equation (1). $\Upsilon(\tau)$ is the two dimensional space-like surface of constant τ, embedded in Σ.

Any linear combination of the primary constraints can be added to (11) without affecting the dynamics. Thus

$$\tilde{H}(\tau) = H_c(\tau) + \int_{\Upsilon(\tau)} d^2\sigma(uK + u^i K_i + u^{ij}K_{ij}) \tag{12}$$

where u, u^i and u^{ij} are arbitrary functions of the canonical variables, is an equally good Hamiltonian. $\tilde{H}(\tau)$ is termed the effective Hamiltonian in [10] and the the total Hamiltonian in [11].

Immediately we see that, in passing to the quantum theory, there will be problems with the operator ordering due to the fact that M contains x^μ, and so will not commute with π_μ. One possible approach is to Weyl order and replace $\pi_\mu(M^{-1})^{\mu\nu}\pi_\nu$ with

$(1/2)(\pi_\mu \pi_\nu (M^{-1})^{\mu\nu} + (M^{-1})^{\mu\nu} \pi_\mu \pi_\nu)$, but this eventually proves inadequate. For the moment the best that can be done, without getting into horrendous algebraic complexities, is to say that any extra terms introduced into the calculation by the non-commutativity of π_μ and x^μ in the quantum theory will involve extra powers of $\hbar$, and so we get at least a first approximation to the quantum theory by ignoring them.

Following Dirac, [11], we must demand that the time evolution of the constraints (10) vanish weakly for consistency

$$\dot{K}(\sigma) = \{K(\sigma), \tilde{H}\} = -\frac{\delta \tilde{H}}{\delta N(\sigma)} \approx -\frac{\delta H_c}{\delta N(\sigma)} \approx 0$$

$$\dot{K}_i(\sigma) \approx -\frac{\delta H_c}{\delta N^i(\sigma)} \approx 0 \qquad \dot{K}_{ij}(\sigma) \approx -\frac{\delta H^c}{\delta h^{ij}(\sigma)} \approx 0.$$

($\{\ ,\ \}$ denote Poisson brackets.)

We find the following six secondary constraints

$$\chi(\sigma) = \frac{1}{2\sqrt{h}}(\pi M^{-1} \pi) + \frac{\sqrt{h}}{4}(\partial_i x M \partial_j x) h^{ij} \approx 0$$

$$\chi_i(\sigma) = (\pi \partial_i x) \approx 0$$

$$\begin{aligned} \chi_{ij}(\sigma) = {} & \frac{N}{4}\left(\frac{\sqrt{h}}{2}(\partial_k x M \partial_l x) h^{kl} - \frac{1}{\sqrt{h}}(\pi M^{-1} \pi)\right) h_{ij} \\ & + \frac{N}{\sqrt{h}}(\pi M^{-1})^\mu M_{ij},_{\mu\nu} (M^{-1}\pi)^\nu - \frac{N}{2}\sqrt{h}(\partial_i x M \partial_j x) \\ \approx {} & 0 \end{aligned}$$

where

$$M_{ij},_{\mu\nu} = G_{ij} g_{\mu\nu} - \partial_i x^\rho \partial_j x^\tau g_{\rho\mu} g_{\tau\nu}$$

The secondary constraints

$$\chi \approx 0 \qquad \chi_i \approx 0 \qquad \chi_{ij} \approx 0$$

are classically equivlent to the algebraic Lagrangian equations of motion $G_{ab} = H_{ab}$ and so, as operator equations in the quantum theory, they are equivalent, at least to lowest order in $\hbar$.

Note that

$$H_c(\tau) = \int_{\Upsilon(\tau)} d^2\sigma (N\chi + N^i \chi_i)$$

is weakly vanishing, as expected on general grounds [11].

Demanding that the time evolution of the χ's vanish weakly gives no new constraints, but merely determines the u's in equation (12).

In the quantum theory, the distinction between primary and secondary constraints ceases to be important, and a more relevant classification is that of first and second class constraints. The Poisson brackets of first class constraints give rise to linear combinations of first class constraints and are weakly vanishing, whereas the Poisson brackets of second class constraints do not vanish, even weakly.

Calculating the Poisson brackets of the twelve constraints (K's and χ's) with each other (and making liberal use of the Lagrangian version of the constraints, (7), *after* the evaluation of their Poisson brackets) gives the following weakly non-zero brackets (after some tedious algebra)

$$\{K_{ij}(\sigma), \chi_{kl}(\sigma')\} \approx N\sqrt{h}\left(h_{ik}h_{jl} + h_{il}h_{jk} - \frac{2}{3}h_{ij}h_{kl}\right)\delta^{(2)}(\sigma - \sigma')$$

$$\{\chi(\sigma), \chi_{ij}(\sigma')\} \approx \frac{2}{3}N\pi_\mu\left(\frac{1}{3}X^\mu_{kl}h^{kl}h_{ij} - X^\mu_{ij}\right)\delta^{(2)}(\sigma - \sigma')$$

$$\{\chi_{ij}(\sigma), \chi_{kl}(\sigma')\} \approx -(N^2/9)\pi_\mu(X^\mu_{ij}h_{kl} - X^\mu_{kl}h_{ij})\delta^{(2)}(\sigma - \sigma')$$

$$\{\chi_{ij}(\sigma), \chi_k(\sigma')\} \approx -2N\sqrt{h}\left(h_{ik}\partial_j\delta^{(2)}(\sigma - \sigma') + h_{jk}\partial_i\delta^{(2)}(\sigma - \sigma') - (2/3)h_{ij}\partial_k\delta^{(2)}(\sigma - \sigma')\right)$$
$$- 2N\sqrt{h}\left(\partial_k h_{ij} - \frac{1}{3h}(\partial_k h)h_{ij}\right)\delta^{(2)}(\sigma - \sigma')$$

where $X^\mu_{ij} \equiv \partial_i\partial_j x^\mu + \partial_i x^\nu \partial_j x^\rho \Gamma^\mu_{\nu\rho}$, $h = det(h_{ij}), \partial_i = \frac{\partial}{\partial\sigma^i}$, and all functions on the right hand sides are evaluated at σ. All other Poisson brackets among the constraints are weakly vanishing. Note in passing that it makes no difference whether the ordinary, (∂_i), or co-variant,$(_{\parallel i})$, derivative is used on $\partial_j x^\mu$ in the definition of X^μ_{ij} above since these are weakly equivalent when contracted with π_μ

$$\pi_\mu(\partial_j x^\mu)_{\parallel i} = \pi_\mu(\partial_i\partial_j x^\mu - \gamma^k_{ij}\partial_k x^\mu) \approx \pi_\mu\partial_i\partial_j x^\mu.$$

However, the term involving $\Gamma^\mu_{\nu\rho}$ cannot be omitted in the definition of X^μ_{ij}.

It is still necessary to disentangle which linear combinations of constraints are first class and which are second class. Following [12], define

$$\tilde{\chi} = \chi + \left(\frac{1}{3\sqrt{h}}\right)K_{ij}X^{ij,\mu}\pi_\mu$$

$$\tilde{\chi}_i = \chi_i + K_{jk}\partial_i h^{jk} + 2\partial_k(K_{ij}h^{jk})$$

$$\tilde{\chi}_{ij} = \chi_{ij} - \left(\frac{N}{18\sqrt{h}}\right)h_{ij}K_{kl}X^{kl,\mu}\pi_\mu$$

(indices i, j, k and l are raised with the two dimensional metric, h^{ij}).

With these definitions $K, K_i, \tilde{\chi}$ and $\tilde{\chi}_i$ are first class, and

$$\{K_{ij}(\sigma), \tilde{\chi}_{kl}(\sigma')\} \approx N\sqrt{h}(h_{ik}h_{jl} + h_{il}h_{jk} - (2/3)h_{ij}h_{kl})\delta^{(2)}(\sigma - \sigma')$$

It is convenient to eliminate the term with $-\frac{2}{3}$, by defining

$$\chi^*_{kl} = \tilde{\chi}_{kl} + (h^{ij}\tilde{\chi}_{ij})h_{kl}$$

giving

$$\begin{aligned}\{K_{ij}(\sigma), \chi^*_{kl}(\sigma')\} &\approx N\sqrt{h}(h_{ik}h_{jl} + h_{il}h_{jk})\delta^{(2)}(\sigma - \sigma') \\ &\equiv C_{ij,kl}(\sigma,\sigma')\end{aligned}$$

Apart from the infinite rank due to σ, $C_{ij,kl}$ is an antisymmetric 6×6 matrix. It has an inverse

$$(C^{-1})^{ij,kl}(\sigma,\sigma') = \frac{1}{4N\sqrt{h}}(h^{ik}h^{jl} + h^{il}h^{jk})\delta^{(2)}(\sigma - \sigma')$$

so that

$$\int_{\Upsilon(\tau)} d^2\sigma C_{ij,kl}(\sigma,\sigma'')(C^{-1})^{kl,i'j'}(\sigma'',\sigma') = (1/2)(\delta_i^{i'}\delta_j^{j'} + \delta_i^{j'}\delta_j^{i'})\delta^{(2)}(\sigma - \sigma').$$

Gauge Conditions

To proceed, the first class constraints must be eliminated by suitable gauge choices, making all constraints second class, [10]. The obvious choices $N = 1$ and $N^i = 0$, effectively eliminate K and K_i, leaving only $\tilde{\chi}$ and $\tilde{\chi}_i$ as first class constraints. A further three constraints (gauge conditions) on x^μ and π_μ are required to turn $\tilde{\chi}$ and $\tilde{\chi}_i$ into second class constraints. The authors of [12] follow the string theorists and fix the center of mass and momentum of the membrane, in the light cone gauge. While these two gauge conditions fix the gauge uniquely for strings, they are not sufficient for membranes. The third degree of freedom is left unfixed in [12]. We shall choose a different gauge, the 'co-ordinate gauge'.

$$x^0 = \tau \qquad x^1 = \sigma^1 \qquad x^2 = \sigma^2 \tag{13}$$

leaving only the transverse co-ordinates, x^m, $m = 3, 4, \ldots, D-1$ as degrees of freedom. The three constraints $\tilde{\chi}$ and $\tilde{\chi}_i$ can now be used to eliminate π_0, π_1 and π_2.

We must be careful that the co-ordinates τ, σ^1 and σ^2 are nowhere singular on Σ. It may be necessary to choose more than one co-ordinate patch on Σ and then match them together on the overlaps using a gauge transformation. These problems can be avoided by restricting ourselves to world volumes, Σ, which can be covered by a single co-ordinate patch. e.g. an open membrane with $-\infty < \tau < \infty$, $0 \leq \sigma^1, \sigma^2 \leq \pi$ with topology $\mathbb{R} \times [0,\pi] \times [0,\pi]$ (though the metric on σ need not be flat). However, this restriction to contractible Σ does not seem to be necessary, provided suitable gauge transformations are used on the co-ordinate overlaps.

First write the gauge conditions as constraints

$$\phi^0 = x^0 - \tau \qquad \phi^1 = x^1 - \sigma^1 \qquad \phi^2 = x^2 - \sigma^2$$

then this gauge is enforced by imposing the conditions

$$\phi^a \approx 0 \qquad (a = 0, 1, 2).$$

The matrix of Poisson brackets of constraints, C, must now be enlarged to include ϕ^a. We need

$$\begin{aligned}
\{\phi^a(\sigma), \phi^b(\sigma')\} &\approx 0 \\
\{\phi^a(\sigma), \tilde{\chi}(\sigma')\} &\approx \frac{1}{6\sqrt{h}}\pi_\mu g^{\mu a}\delta^{(2)}(\sigma - \sigma\prime) \\
\{\phi^a(\sigma), \tilde{\chi}_j(\sigma')\} &\approx \delta^a_j \delta^{(2)}(\sigma - \sigma') \\
\{\phi^a(\sigma) \chi^*_{ij}(\sigma')\} &\approx -\frac{1}{12\sqrt{h}} h_{ij}\pi_\mu g^{\mu a}\delta^{(2)}(\sigma - \sigma').
\end{aligned}$$

C now becomes an antisymmetric 12×12 matrix $(s, s' = 1, \ldots, 12)$

$$C_{s,s'} = \begin{array}{c} \phi^0 \\ \phi^1 \\ \phi^2 \\ \tilde\chi \\ \tilde\chi_1 \\ \tilde\chi_2 \\ K_{11} \\ K_{12} \\ K_{22} \\ \chi^*_{11} \\ \chi^*_{12} \\ \chi^*_{22} \end{array} \overset{\phi^0\ \phi^1\ \phi^2\ \tilde\chi\ \tilde\chi_1\ \tilde\chi_2\ K_{11}\ K_{12}\ K_{22}\ \chi^*_{11}\ \chi^*_{12}\ \chi^*_{22}}{\begin{pmatrix} \mathbf{0} & \mathbf{R} & \mathbf{0} & -\mathbf{Q}^T \\ -\mathbf{R}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{\mathbf{h}} \\ \mathbf{Q} & \mathbf{0} & -\tilde{\mathbf{h}} & \mathbf{0} \end{pmatrix}} \delta^{(2)}(\sigma-\sigma')$$

where the 3×3 matrices $\mathbf{R}$, $\mathbf{Q}$ and $\tilde{\mathbf{h}}$ are defined as

$$\mathbf{R} = \begin{pmatrix} \pi^0/(6\sqrt{h}) & 0 & 0 \\ \pi^1/(6\sqrt{h}) & 1 & 0 \\ \pi^2/(6\sqrt{h}) & 0 & 1 \end{pmatrix}$$

$$\mathbf{Q}_{ij,}{}^{a} = \frac{1}{12\sqrt{h}}(h_{ij}\pi_\mu g^{\mu a}) \qquad \text{and} \qquad \tilde{\mathbf{h}}_{ij,kl} = \sqrt{h}(h_{ik}h_{jl} + h_{il}h_{jk}).$$

For example

$$C_{1,4'}(\sigma,\sigma') = \frac{\pi^0}{6\sqrt{h}}\delta^{(2)}(\sigma-\sigma') \approx \{\phi^0(\sigma), \tilde\chi(\sigma')\}.$$

The inverse matrix $(C^{-1})^{s,s'}$ is readily obtained

$$(C^{-1})^{s,s'}(\sigma,\sigma') = \begin{pmatrix} \mathbf{0} & -(\mathbf{R}^T)^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{R}^{-1} & \mathbf{0} & \mathbf{S} & \mathbf{0} \\ \mathbf{0} & -\mathbf{S}^T & \mathbf{0} & -\tilde{\mathbf{h}}^{-1} \\ \mathbf{0} & \mathbf{0} & \tilde{\mathbf{h}}^{-1} & \mathbf{0} \end{pmatrix} \delta^{(2)}(\sigma-\sigma')$$

with

$$\mathbf{R}^{-1} = \begin{pmatrix} (6\sqrt{h})/\pi^0 & 0 & 0 \\ -(\pi^1/\pi^0) & 1 & 0 \\ -(\pi^2/\pi^0) & 0 & 1 \end{pmatrix} \qquad \mathbf{S} = \begin{pmatrix} h^{11}/(4\sqrt{h}) & h^{12}/(4\sqrt{h}) & h^{22}/(4\sqrt{h}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$(\tilde{\mathrm{h}}^{-1})^{ij,kl} = \frac{1}{4\sqrt{h}}(h^{ik}h^{jl} + h^{il}h^{jk}).$$

The commutators of the remaining canonical variables, x^m and π_m, in the quantum theory are now obtained from the Dirac brackets, [11]

$$\{A(\sigma), B(\sigma')\}^* = \{A(\sigma), B(\sigma')\}$$
$$- \int\int_{\Upsilon(\tau)} d^2\sigma_1 d^2\sigma_2 \{A(\sigma), \phi_s(\sigma_1)\}(C^{-1})^{s,s'}(\sigma_1, \sigma_2)\{\phi_{s'}(\sigma_2), B(\sigma')\}$$

where ϕ_s represent the twelve second class constraints.

We find quite simply

$$\{x^m(\sigma), \pi_n(\sigma')\}^* \approx \delta^m_n \delta^{(2)}(\sigma - \sigma')$$

i.e. the degrees of freedom transverse to the world volume of the membrane are unaffected by the presence of the constraints.

Of course, the Dirac brackets of π_a and x^a will be more complicated, but these are no longer free canonical variables, due to the constraints ϕ^a, $\tilde{\chi}$ and $\tilde{\chi}_i$. Explicitly

$$\phi^a \quad \text{and} \quad \tilde{\chi}_1 \approx 0 \qquad \Rightarrow \qquad \pi_1 \approx -\pi_m \partial_1 x^m$$

$$\phi^a \quad \text{and} \quad \tilde{\chi}_2 \approx 0 \qquad \Rightarrow \qquad \pi_2 \approx -\pi_m \partial_2 x^m$$

and $\tilde{\chi} \approx 0$ can be used to find an expression for π_0 in terms of x^m and π_m, but we do not write it down because it is not not very illuminating.

Finally, the extended Hamiltonian, [11], is obtained by adding linear combinations of $(K, K_i, \tilde{\chi}, \tilde{\chi}_i) \equiv (\psi_A)$ to (12)

$$H_E(\tau) = \tilde{H}(\tau) + \int_{\Upsilon(\tau)} d^2\sigma v^A(\sigma)\psi_A(\sigma) \qquad A = 1, \ldots, 6.$$

The functions v^A are determined by demanding consistent time evolution of the gauge constraints.

Conclusions

It has been shown that, with the action for a bosonic membrane moving in D dimensions given by (5), the Hamiltonian (11) can be quantised, at least to lowest order in $\hbar$, using Dirac's method for systems with constraints. In the co-ordinate gauge,

$$x^0 = \tau \qquad x^1 = \sigma^1 \qquad x^2 = \sigma^2$$

the canonical degrees of freedom $\{x^m, \pi_m\}$ $(m, n = 3, 4, \ldots, D)$ satisfy Dirac brackets

$$\{x^m(\sigma), \pi_n(\sigma')\}^* = \delta^m_n \delta^{(2)}(\sigma - \sigma')$$

at least to lowest order in $\hbar$.

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