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LIMIT THEOREMS FOR STOCHASTIC PROCESSES ASSOCIATED WITH A BOSON GAS

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§1 INTRODUCTION

In this lecture, we discuss the density of particles having energy less than λ in a boson system as a stochastic process indexed by λ . The notation is that of [1] in this volume. Recall that the hamiltonian for the free boson gas is given by

$$H_{\lambda}(\omega) = \sum_{j \geq 1} \lambda_{\lambda}(j) \sigma_j(\omega), \quad (1.1)$$

where $0 = \lambda_{\lambda}(1) \leq \lambda_{\lambda}(2) \leq \dots$. For a system in a region of volume V_{λ} , the grand canonical pressure $p_{\lambda}(\mu)$ is defined for $\mu < 0$ by

$$p_{\lambda}(\mu) = \frac{1}{\beta V_{\lambda}} \ln \left\{ \sum_{\omega \in \Omega} e^{\beta(\mu N(\omega) - H_{\lambda}(\omega))} \right\}. \quad (1.2)$$

In [2] in this volume, we recalled results (proved in [3]) on the existence of the pressure in the thermodynamic limit:

$$p(\mu) = \lim_{\lambda \rightarrow \infty} p_{\lambda}(\mu). \quad (1.3)$$

In order to discuss the phenomenon of boson condensation, we introduced in [3] the family of random variables $\{X_{\lambda}(\cdot; \lambda) : \lambda \geq 0\}$ defined by

$$X_{\lambda}(\omega; \lambda) = \frac{1}{V_{\lambda}} \sum_{\{j : \lambda_{\lambda}(j) \leq \lambda\}} \sigma_j(\omega). \quad (1.4)$$

For the free boson gas, we have the following result:

THEOREM 1

Suppose that (S1) and (S2) hold; then, for ρ_C finite.

$$\lim_{\lambda \downarrow 0} \lim_{\lambda \rightarrow \infty} \mathbb{E}_{\lambda}^{\rho} [X_{\lambda}(\lambda)] = (\rho - \rho_C)^+. \quad (1.5)$$

[Conditions (S1) and (S2) and the critical density ρ_C are defined in §2 and §3 of [2] in this volume. Here $\mathbb{E}_{\lambda}^{\rho}[\cdot]$ denotes the expectation taken with respect to the grand canonical probability measure $\mathbb{P}_{\lambda}^{\mu}[\cdot]$ with $\mu = \mu_{\lambda}(\rho)$, defined in §3 of

[2]; it is the expectation at fixed mean density ρ .]

Proof:

From the definition of $X_\ell(\lambda)$, we have

$$\mathbb{E}_\ell^\rho[X_\ell(\lambda)] = \int_{[0, \lambda)} p'(\mu_\ell(\rho)|\lambda) dF_\ell(\lambda) = \rho - \int_{[\lambda, \infty)} p'(\mu_\ell(\rho)|\lambda) dF_\ell(\lambda). \quad (1.6)$$

But, for $\mu < \lambda$, the sequence

$$\left\{ \int_{[\lambda, \infty)} p'(\mu|\lambda) dF_\ell(\lambda) : \ell = 1, 2, \dots \right\} \quad (1.7)$$

converges uniformly in μ on compacts to

$$\int_{[\lambda, \infty)} p'(\mu|\lambda) dF(\lambda). \quad (1.8)$$

Hence, by Proposition 2 of [2], we have for $\lambda > 0$:

$$\lim_{\ell \rightarrow \infty} \mathbb{E}_\ell^\rho[X_\ell(\lambda)] = \rho - \int_{[\lambda, \infty)} p(\mu(\lambda)|\lambda) dF(\lambda). \quad (1.9)$$

But, by hypothesis, ρ_C is finite so that we may invoke the dominated convergence principle to conclude that

$$\lim_{\lambda \downarrow 0} \int_{[\lambda, \infty)} p'(\mu(\rho)|\lambda) dF(\lambda) = \int_{[0, \infty)} p'(\mu(\rho)|\lambda) dF(\lambda) = \begin{cases} \rho & , \rho < \rho_C, \\ \rho_C & , \rho \geq \rho_C. \end{cases}$$

Thus we have

$$\lim_{\ell \rightarrow \infty} \mathbb{E}_\ell^\rho[X_\ell(\lambda)] = (\rho - \rho_C)^+ \quad \blacksquare$$

In the free boson gas there is a second effect, discovered by M. Kac in 1971. We saw in §3 of [2] that the free-energy has a first-order phase-transition segment $[\rho_C, \infty)$; it follows that for $\rho > \rho_C$ there is no guarantee that the weak law of large numbers will hold for the distribution $\mathbb{K}_\ell^\rho = \mathbb{P}_\ell^\rho \circ X_\ell^{-1}$ of the number density $X_\ell = N/V_\ell$. In fact, there is no guarantee that for, $\rho > \rho_C$, the sequence $\{\mathbb{K}_\ell^\rho : \ell = 1, 2, \dots\}$ will converge; nevertheless, by the Helly Selection Principle, a subsequence will converge, but the limit distribution will depend on the detailed behaviour of the corresponding subsequence of the sequence $\{\lambda_\ell(\cdot) : \ell = 1, 2, \dots\}$. In other words, it is possible to have two sequences, $\{\lambda_\ell(\cdot) : \ell = 1, 2, \dots\}$ and $\{\hat{\lambda}_\ell(\cdot) : \ell = 1, 2, \dots\}$, each satisfying (S1) and (S2) and having the same integrated density of states $F(\cdot)$ but having limit distributions \mathbb{K}^ρ and $\hat{\mathbb{K}}^\rho$ which are

distinct for $\rho > \rho_C$. (For $\rho < \rho_C$, they must both be equal to δ_ρ , the degenerate distribution concentrated at ρ , by Theorem 1 of [1].) For example, Kac showed that in the standard example (described in §3 of [2]) the limit distribution is the exponential distribution supported on $[\rho_C, \infty)$ with mean ρ , for $\rho > \rho_C$; other examples are investigated in detail in [3]. We shall see in the next section that, in the mean-field model, this phenomenon disappears: there is no first-order phase-transition segment, the grand canonical pressure exists for all values of μ and is a differentiable function; the weak law of large numbers holds for X_ℓ for all values of the mean density ρ ; nevertheless, condensation persists. In these circumstances it becomes interesting to regard $\lambda \rightarrow X_\ell(\cdot; \lambda)$ as a stochastic process and to enquire about the convergence in distribution of a re-scaled, centred version of it. This we do in §3.

§2 THE MEAN FIELD-MODEL

To describe the mean-field model, we define a sequence of hamiltonians $\{\tilde{H}_\ell; \ell = 1, 2, \dots\}$ by

$$\tilde{H}_\ell(\omega) = H_\ell(\omega) + \frac{a}{2V_\ell} N^2(\omega) \quad (2.1)$$

with $a > 0$. The term $\frac{a}{2V_\ell} N^2$, which provides a crude caricature of the interaction,

can be understood classically: it arises in an "index of refraction" approximation in which we imagine each particle to move through the system as if it were moving in a uniform optical medium and so receiving an increment of energy proportional to the density $X_\ell = N/V_\ell$; since a is positive the interaction is repulsive.

First, we compute the pressure $\tilde{p}_\ell(\mu)$, as explained in §4 of [1]: writing $u(x) = (\mu - \alpha)x - \frac{a}{2}x^2$, a straight-forward manipulation gives

$$\tilde{p}_\ell(\mu) = p_\ell(\alpha) + \frac{1}{\beta V_\ell} \ln \mathbb{E}_\ell^\alpha [e^{\beta V_\ell u(X_\ell)}] = p_\ell(\alpha) + \frac{1}{\beta V_\ell} \ln \int_{[0, \infty)} e^{\beta V_\ell u(x)} K_\ell^\alpha [dx] \quad (2.2)$$

for each $\alpha < 0$, where $K_\ell^\alpha = P_\ell^\alpha \circ X_\ell^{-1}$. But $x \rightarrow u(x)$ is continuous and bounded above and $\{K_\ell^\alpha; \ell = 1, 2, \dots\}$ satisfies the Large Deviation Principle with rate-function $I^\alpha(x) = p(\alpha) + f(x) - \alpha x$, by Theorem 1 of [2]. Hence, by Varadhan's First Theorem, $\tilde{p}(\mu) = \lim_{\ell \rightarrow \infty} \tilde{p}_\ell(\mu)$ exists and is given by

$$\tilde{p}(\mu) = p(\alpha) + \sup_x \{u(x) - I^\alpha(x)\} = \sup_x \{\mu x - \tilde{f}(x)\}, \quad (2.3)$$

where the mean-field free-energy $\tilde{f}(\cdot)$ is given by

$$\tilde{f}(x) = f(x) + \frac{a}{2}x^2. \quad (2.4)$$

Thus we have proved:

THEOREM 2

Suppose that (S1) and (S2) hold; then the mean-field pressure exists for all real μ and is given by

$$\tilde{p}(\mu) = \sup_x \{\mu x - \tilde{f}(x)\}, \quad (2.5)$$

where $x \rightarrow \tilde{f}(x)$ is the mean-field free energy, given by $\tilde{f}(x) = f(x) + \frac{a}{2}x^2$.

Next, we introduce the mean-field expectation functional $\tilde{E}_l^\mu[\cdot]$ defined by

$$\tilde{E}_l^\mu[\cdot] = \mathbb{E}_l^\alpha[e^{\beta M_l}] / \mathbb{E}_l^\alpha[e^{\beta M_l}], \quad (2.6)$$

and the associated probability measure $\tilde{P}_l^\alpha[\cdot]$, where

$$M_l = V_l u(X_l). \quad (2.7)$$

COROLLARY

The mean-field pressure $\mu \rightarrow \tilde{p}(\mu)$ is differentiable for all values of μ . The sequence of distribution functions $\{\tilde{K}_l^\mu = \tilde{P}_l^\mu \circ X_l^{-1}\}$ converges weakly to the degenerate distribution δ_ρ concentrated at $\rho = \tilde{p}'(\mu)$ and satisfies the Large Deviation Principle with constants $\{V_l\}$ and rate-function $\tilde{I}^\mu(x) = \tilde{p}(\mu) + \tilde{f}(x) - \mu x$.

Proof:

Since $x \rightarrow f(x)$ is strictly convex for $0 \leq x < \rho_c$ and constant for $\rho_c \leq x < \infty$ and $x \rightarrow \frac{a}{2}x^2$ is strictly convex for $0 \leq x < \infty$, the function $x \rightarrow \tilde{f}(x) = f(x) + \frac{a}{2}x^2$ is strictly convex for $0 \leq x < \infty$; hence there is no first-order phase-transition segment; equivalently, $\mu \rightarrow \tilde{p}(\mu)$, the Legendre transform of $x \rightarrow \tilde{f}(x)$, is differentiable for $\mu < \infty$. It follows from Theorem 1 of [1] that $\tilde{K}_l^\mu \rightarrow \delta_\rho$, where $\rho = \tilde{p}'(\mu)$, and from Theorem 4 of [1] that $\{\tilde{K}_l^\mu; l = 1, 2, \dots\}$ satisfies the Large Deviation Principle with constants $\{V_l\}$ and rate-function $\tilde{I}^\mu(\cdot)$ ■

Although the first-order phase-transition segment, which was present in the free energy function of the free-gas, has disappeared, the phenomenon of condensation persists:

THEOREM 3

Suppose that (S1) and (S2) hold; then, for ρ_C finite, we have

$$\lim_{\lambda \downarrow 0} \lim_{\ell \rightarrow \infty} \tilde{\mathbb{E}}_{\ell}^{\mu}[X_{\ell}(\lambda)] = (\rho - \rho_C)^+, \quad (2.8)$$

where $\tilde{\mathbb{E}}_{\ell}^{\mu}[\cdot]$ is the mean-field expectation functional and $\rho = \tilde{p}'(\mu)$.

Proof:

First, we remark that an elementary exercise yields the following alternative formula for the mean-field pressure $\tilde{p}(\mu)$:

$$\tilde{p}(\mu) = \inf_{\alpha < 0} \left\{ \frac{(\mu - \alpha)^2}{2a} + p(\alpha) \right\}, \quad (2.9)$$

where $p(\alpha)$ is the free-gas pressure. The idea of the proof of (2.8) is that we compute the cumulant generating function of $X_{\ell}(\lambda)$; since

$$V_{\ell} X_{\ell}(\lambda) = V_{\ell} X_{\ell} - \sum_{\{j: \lambda_{\ell}(j) > \lambda\}} \sigma_j \quad (2.10)$$

we get

$$\tilde{\mathbb{E}}_{\ell}^{\mu} [e^{\beta s V_{\ell} X_{\ell}(\lambda)}] = \tilde{\mathbb{E}}_{\ell}^{(s, \lambda), \mu} [e^{\beta s V_{\ell} X_{\ell}}],$$

where $\tilde{\mathbb{E}}_{\ell}^{(s, \lambda), \mu}[\cdot]$ is the mean-field expectation functional for which the free-gas hamiltonian has been modified by the addition of the term $\sum_{\{j: \lambda_{\ell}(j) > \lambda\}} s \sigma_j$. These considerations yield the formula

$$\lim_{\ell \rightarrow \infty} \tilde{\mathbb{E}}_{\ell}^{\mu}[X_{\ell}(\lambda)] = \left. \frac{\partial}{\partial s} \tilde{p}(\mu + s; s, \lambda) \right|_{s=0}, \quad (2.11)$$

where

$$\tilde{p}(\mu + s; s, \lambda) = \inf_{\alpha < 0} \left\{ \frac{(\mu + s - \alpha)^2}{2a} + p(\alpha; s, \lambda) \right\},$$

and

$$p(\alpha; s, \lambda) = \int_{(0, \lambda)} p(\alpha | \lambda) dF(\lambda) + \int_{[\lambda, \infty)} p(\alpha | s + \lambda) dF(\lambda).$$

A standard argument, using Griffith's Lemma, yields the result. ■

§3 FLUCTUATIONS IN THE MEAN-FIELD MODEL

Fluctuations in $X_{\ell} = N/V_{\ell}$ in the mean-field model in the thermodynamic limit

were studied for the standard example, described in §3 of [2] in this volume, by Davies [4], Wreszinski [5], Fannes and Verbeure [6] and Buffet and Pule [7]. The mean-field model in the general situation, where the only assumptions about the single-particle spectrum are that (S1) and (S2) hold, was investigated in [8]; we have summarized the results of [8] in §2 and now go on to investigate the fluctuations in X_μ . In fact, we do rather more; we regard $\lambda \rightarrow X_\mu(\lambda)$ as a stochastic process and prove a central limit theorem:

THEOREM 4

Let $Z_\mu(\lambda) = V_\mu^{1/2} \{X_\mu(\lambda) - \tilde{\mathbb{E}}_\mu[X_\mu(\lambda)]\}$; then, for $\mu < ap_C$, $Z_\mu(\lambda) \xrightarrow{(d)} Z(\lambda)$, where $Z(\lambda)$ is gaussian with mean zero and covariance $\Gamma(\lambda_1, \lambda_2)$ given by

$$\Gamma(\lambda_1, \lambda_2) = J_{\lambda_1 \wedge \lambda_2}^\mu - \frac{aJ_{\lambda_1}^\mu J_{\lambda_2}^\mu}{1 + aJ_\infty^\mu} \quad (3.1)$$

where

$$J_\lambda^\mu = \int_{[0, \lambda)} p''(\alpha(\mu) | \lambda) dF(\lambda) \quad (3.2)$$

and $\alpha(\mu)$ is the value of α at which $\inf_{\alpha < 0} \left\{ \frac{(\mu - \alpha)^2}{2a} + p(\alpha) \right\}$ is attained.

Sketch of proof:

The result follows from a routine, but somewhat tedious, calculation of

$$\lim_{\mu \rightarrow \infty} \tilde{\mathbb{E}}_\mu^\mu \left[e^{\beta(s_1 Z_\mu(\lambda_1) + s_2 Z_\mu(\lambda_2))} \right]$$

along the lines of the proof of Theorem 3. ■

It is interesting to identify the process $Z(\cdot)$ in terms of a standard process.

THEOREM 5

Let $B(\cdot)$ be a BM(1), a brownian motion in \mathbb{R}^1 starting at zero; then, for $\mu < ap_C$

$$Z(\lambda) \stackrel{(d)}{=} B(J_\lambda^\mu) - \frac{aJ_\lambda^\mu}{1 + aJ_\infty^\mu} B(J_\infty^\mu + 1/a). \quad (3.3)$$

Proof:

A routine computation shows that the mean of the right-hand side of (3.3) is zero and the covariance is the same as that of $Z(\cdot)$, given by (3.1). Hence the two gaussian processes are equal in distribution. ■

The process (3.3) is a modification of a time-changed brownian bridge; it never reaches the point at which it is tied-down but, as a increases, that point comes closer to $J_{\frac{\mu}{a}}$. This shows how, as the strength of the interaction increases, the fluctuations in $Z(\infty)$ are damped down.

It is a little more difficult to deal with the case $\mu > ap_C$; we introduce

$$W_g(\lambda) = Z_g(\infty) - Z_g(\lambda) \quad (3.4)$$

and prove in analagous fashion:

THEOREM 6

For $\mu > ap_C$, $W_g(\lambda) \xrightarrow{(d)} W(\lambda)$, where $W(\lambda)$ is a gaussian process with mean zero and covariance $\Gamma(\lambda_1, \lambda_2)$ given by

$$\Gamma(\lambda_1, \lambda_2) = K_{\lambda_1}^{\mu} \vee \lambda_2, \quad (3.5)$$

where

$$K_{\lambda}^{\mu} = \int_{[\lambda, \infty)} p''(0|\lambda) dF(\lambda). \quad (3.6)$$

In this case,

$$W(\lambda) \stackrel{(d)}{=} K_{\lambda}^{\mu} B\left[\frac{1}{K_{\lambda}^{\mu}}\right]. \quad (3.7)$$

The method by which we discovered the representations may be of some interest. The stochastic differential equation satisfied by a process $(X_t)_{t \geq 0}$ with filtration (\mathcal{F}_t) is discussed by Nelson [9]; see also McGill [10].

Suppose that a process (X_t, \mathcal{F}_t) satisfies the stochastic differential equation

$$X_t = X_s + \int_s^t \sigma(u, X_u) dB(u) + \int_s^t \tau(u, X_u) du, \quad (3.8)$$

then

$$\tau(s, X_s) = \lim_{t \downarrow s} \frac{1}{t-s} \mathbb{E} [X_t - X_s | \mathcal{F}_s] \quad (3.9)$$

and

$$\sigma^2(s, X_s) = \lim_{t \downarrow s} \frac{1}{t-s} \mathbb{E} [X_t - X_s]^2 | \mathcal{F}_s]. \quad (3.10)$$

Assuming that the processes $Z(\lambda), W(\lambda)$ satisfy stochastic differential equations, the corresponding coefficients σ and τ can be computed using (3.9) and (3.10); this is a routine exercise starting from the expressions (3.1) and (3.5) for the covariances since the processes are gaussian. Obvious time-changes then give the stochastic differential equations for a brownian bridge and a brownian motion respectively.

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§1 INTRODUCTION

In this lecture, we discuss the density of particles having energy less than λ in a boson system as a stochastic process indexed by λ . The notation is that of [1] in this volume. Recall that the hamiltonian for the free boson gas is given by

$$H_\ell(\omega) = \sum_{j \geq 1} \lambda_\ell(j) \sigma_j(\omega), \quad (1.1)$$

where $0 = \lambda_\ell(1) \leq \lambda_\ell(2) \leq \dots$. For a system in a region of volume V_ℓ , the grand canonical pressure $p_\ell(\mu)$ is defined for $\mu < 0$ by

$$p_\ell(\mu) = \frac{1}{\beta V_\ell} \ln \left\{ \sum_{\omega \in \Omega} e^{\beta(\mu N(\omega) - H_\ell(\omega))} \right\}. \quad (1.2)$$

In [2] in this volume, we recalled results (proved in [3]) on the existence of the pressure in the thermodynamic limit:

$$p(\mu) = \lim_{\ell \rightarrow \infty} p_\ell(\mu). \quad (1.3)$$

In order to discuss the phenomenon of boson condensation, we introduced in [3] the family of random variables $\{X_\ell(\cdot; \lambda) : \lambda \geq 0\}$ defined by

$$X_\ell(\omega; \lambda) = \frac{1}{V_\ell} \sum_{\{j : \lambda_\ell(j) \leq \lambda\}} \sigma_j(\omega). \quad (1.4)$$

For the free boson gas, we have the following result:

THEOREM 1

Suppose that (S1) and (S2) hold; then, for ρ_C finite,

$$\lim_{\lambda \downarrow 0} \lim_{\ell \rightarrow \infty} \mathbb{E}_\ell^\rho [X_\ell(\lambda)] = (\rho - \rho_C)^+. \quad (1.5)$$

[Conditions (S1) and (S2) and the critical density ρ_C are defined in §2 and §3 of [2] in this volume. Here $\mathbb{E}_\ell^\rho[\cdot]$ denotes the expectation taken with respect to the grand canonical probability measure $\mathbb{P}_\ell^\mu[\cdot]$ with $\mu = \mu_\ell(\rho)$, defined in §3 of

[2]; it is the expectation at fixed mean density ρ .]

Proof:

From the definition of $X_\ell(\lambda)$, we have

$$\mathbb{E}_\ell^\rho[X_\ell(\lambda)] = \int_{[0, \lambda)} p'(\mu_\ell(\rho)|\lambda) dF_\ell(\lambda) = \rho - \int_{[\lambda, \infty)} p'(\mu_\ell(\rho)|\lambda) dF_\ell(\lambda). \quad (1.6)$$

But, for $\mu < \lambda$, the sequence

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Hence, by Proposition 2 of [2], we have for $\lambda > 0$:

$$\lim_{\ell \rightarrow \infty} \mathbb{E}_\ell^\rho[X_\ell(\lambda)] = \rho - \int_{[\lambda, \infty)} p(\mu(\lambda)|\lambda) dF(\lambda). \quad (1.9)$$

But, by hypothesis, ρ_C is finite so that we may invoke the dominated convergence principle to conclude that

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Thus we have

$$\lim_{\ell \rightarrow \infty} \mathbb{E}_\ell^\rho[X_\ell(\lambda)] = (\rho - \rho_C)^+ \quad \blacksquare$$

In the free boson gas there is a second effect, discovered by M. Kac in 1971. We saw in §3 of [2] that the free-energy has a first-order phase-transition segment $[\rho_C, \infty)$; it follows that for $\rho > \rho_C$ there is no guarantee that the weak law of large numbers will hold for the distribution $\mathbb{K}_\ell^\rho = \mathbb{P}_\ell^\rho \circ X_\ell^{-1}$ of the number density $X_\ell = N/V_\ell$. In fact, there is no guarantee that for, $\rho > \rho_C$, the sequence $\{\mathbb{K}_\ell^\rho : \ell = 1, 2, \dots\}$ will converge; nevertheless, by the Helly Selection Principle, a subsequence will converge, but the limit distribution will depend on the detailed behaviour of the corresponding subsequence of the sequence $\{\lambda_\ell(\cdot) : \ell = 1, 2, \dots\}$. In other words, it is possible to have two sequences, $\{\lambda_\ell(\cdot) : \ell = 1, 2, \dots\}$ and $\{\hat{\lambda}_\ell(\cdot) : \ell = 1, 2, \dots\}$, each satisfying (S1) and (S2) and having the same integrated density of states $F(\cdot)$ but having limit distributions \mathbb{K}^ρ and $\hat{\mathbb{K}}^\rho$ which are

distinct for $\rho > \rho_c$. (For $\rho < \rho_c$, they must both be equal to δ_ρ , the degenerate distribution concentrated at ρ , by Theorem 1 of [1].) For example, Kac showed that in the standard example (described in §3 of [2]) the limit distribution is the exponential distribution supported on $[\rho_c, \infty)$ with mean ρ , for $\rho > \rho_c$; other examples are investigated in detail in [3]. We shall see in the next section that, in the mean-field model, this phenomenon disappears: there is no first-order phase-transition segment, the grand canonical pressure exists for all values of μ and is a differentiable function; the weak law of large numbers holds for X_ℓ for all values of the mean density ρ ; nevertheless, condensation persists. In these circumstances it becomes interesting to regard $\lambda \rightarrow X_\ell(\cdot; \lambda)$ as a stochastic process and to enquire about the convergence in distribution of a re-scaled, centred version of it. This we do in §3.

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$$\tilde{p}_\ell(\mu) = p_\ell(\alpha) + \frac{1}{\beta V_\ell} \ln \mathbb{E}_\ell^\alpha [e^{\beta V_\ell u(X_\ell)}] = p_\ell(\alpha) + \frac{1}{\beta V_\ell} \ln \int_{[0, \infty)} e^{\beta V_\ell u(x)} K_\ell^\alpha [dx] \quad (2.2)$$

for each $\alpha < 0$, where $K_\ell^\alpha = P_\ell^\alpha \circ X_\ell^{-1}$. But $x \rightarrow u(x)$ is continuous and bounded above and $\{K_\ell^\alpha; \ell = 1, 2, \dots\}$ satisfies the Large Deviation Principle with rate-function $I^\alpha(x) = p(\alpha) + f(x) - \alpha x$, by Theorem 1 of [2]. Hence, by Varadhan's First Theorem, $\tilde{p}(\mu) = \lim_{\ell \rightarrow \infty} \tilde{p}_\ell(\mu)$ exists and is given by

$$\tilde{p}(\mu) = p(\alpha) + \sup_x \{u(x) - I^\alpha(x)\} = \sup_x \{\mu x - \tilde{f}(x)\}, \quad (2.3)$$

where the mean-field free-energy $\tilde{f}(\cdot)$ is given by

$$\tilde{f}(x) = f(x) + \frac{a}{2}x^2. \quad (2.4)$$

Thus we have proved:

THEOREM 2

Suppose that (S1) and (S2) hold; then the mean-field pressure exists for all real μ and is given by

$$\tilde{p}(\mu) = \sup_x \{ \mu x - \tilde{f}(x) \}, \quad (2.5)$$

where $x \rightarrow \tilde{f}(x)$ is the mean-field free energy, given by $\tilde{f}(x) = f(x) + \frac{a}{2}x^2$.

Next, we introduce the mean-field expectation functional $\tilde{\mathbb{E}}_l^\mu[\cdot]$ defined by

$$\tilde{\mathbb{E}}_l^\mu[\cdot] = \mathbb{E}_l^\alpha[\cdot e^{\beta M_l}] / \mathbb{E}_l^\alpha[e^{\beta M_l}], \quad (2.6)$$

and the associated probability measure $\tilde{\mathbb{P}}_l^\alpha[\cdot]$, where

$$M_l = V_l u(X_l). \quad (2.7)$$

COROLLARY

The mean-field pressure $\mu \rightarrow \tilde{p}(\mu)$ is differentiable for all values of μ . The sequence of distribution functions $\{\tilde{\mathbb{K}}_l^\mu = \tilde{\mathbb{P}}_l^\mu \circ X_l^{-1}\}$ converges weakly to the degenerate distribution δ_ρ concentrated at $\rho = \tilde{p}'(\mu)$ and satisfies the Large Deviation Principle with constants $\{V_l\}$ and rate-function $\tilde{\mathbb{I}}^\mu(x) = \tilde{p}(\mu) + \tilde{f}(x) - \mu x$.

Proof:

Since $x \rightarrow f(x)$ is strictly convex for $0 \leq x < \rho_c$ and constant for $\rho_c \leq x < \infty$ and $x \rightarrow \frac{a}{2}x^2$ is strictly convex for $0 \leq x < \infty$, the function $x \rightarrow \tilde{f}(x) = f(x) + \frac{a}{2}x^2$ is strictly convex for $0 \leq x < \infty$; hence there is no first-order phase-transition segment; equivalently, $\mu \rightarrow \tilde{p}(\mu)$, the Legendre transform of $x \rightarrow \tilde{f}(x)$, is differentiable for $\mu < \infty$. It follows from Theorem 1 of [1] that $\tilde{\mathbb{K}}_l^\mu \rightarrow \delta_\rho$, where $\rho = \tilde{p}'(\mu)$, and from Theorem 4 of [1] that $\{\tilde{\mathbb{K}}_l^\mu; l = 1, 2, \dots\}$ satisfies the Large Deviation Principle with constants $\{V_l\}$ and rate-function $\tilde{\mathbb{I}}^\mu(\cdot)$ ■

Although the first-order phase-transition segment, which was present in the free energy function of the free-gas, has disappeared, the phenomenon of condensation persists:

THEOREM 3

Suppose that (S1) and (S2) hold; then, for ρ_C finite, we have

$$\lim_{\lambda \downarrow 0} \lim_{\ell \rightarrow \infty} \tilde{\mathbb{E}}_{\ell}^{\mu}[X_{\ell}(\lambda)] = (\rho - \rho_C)^+, \quad (2.8)$$

where $\tilde{\mathbb{E}}_{\ell}^{\mu}[\cdot]$ is the mean-field expectation functional and $\rho = \tilde{p}'(\mu)$.

Proof:

First, we remark that an elementary exercise yields the following alternative formula for the mean-field pressure $\tilde{p}(\mu)$:

$$\tilde{p}(\mu) = \inf_{\alpha < 0} \left\{ \frac{(\mu - \alpha)^2}{2a} + p(\alpha) \right\}, \quad (2.9)$$

where $p(\alpha)$ is the free-gas pressure. The idea of the proof of (2.8) is that we compute the cumulant generating function of $X_{\ell}(\lambda)$; since

$$V_{\ell} X_{\ell}(\lambda) = V_{\ell} X_{\ell} - \sum_{\{j: \lambda_{\ell}(j) > \lambda\}} \sigma_j \quad (2.10)$$

we get

$$\tilde{\mathbb{E}}_{\ell}^{\mu} [e^{\beta s V_{\ell} X_{\ell}(\lambda)}] = \tilde{\mathbb{E}}_{\ell}^{(s, \lambda), \mu} [e^{\beta s V_{\ell} X_{\ell}}],$$

where $\tilde{\mathbb{E}}_{\ell}^{(s, \lambda), \mu}[\cdot]$ is the mean-field expectation functional for which the free-gas hamiltonian has been modified by the addition of the term $\sum_{\{j: \lambda_{\ell}(j) > \lambda\}} s \sigma_j$. These considerations yield the formula

$$\lim_{\ell \rightarrow \infty} \tilde{\mathbb{E}}_{\ell}^{\mu}[X_{\ell}(\lambda)] = \left. \frac{\partial}{\partial s} \tilde{p}(\mu + s; s, \lambda) \right|_{s=0}, \quad (2.11)$$

where

$$\tilde{p}(\mu + s; s, \lambda) = \inf_{\alpha < 0} \left\{ \frac{(\mu + s - \alpha)^2}{2a} + p(\alpha; s, \lambda) \right\},$$

and

$$p(\alpha; s, \lambda) = \int_{[0, \lambda)} p(\alpha | \lambda) dF(\lambda) + \int_{[\lambda, \infty)} p(\alpha | s + \lambda) dF(\lambda).$$

A standard argument; using Griffith's Lemma, yields the result. ■

§3 FLUCTUATIONS IN THE MEAN-FIELD MODEL

Fluctuations in $X_{\ell} = N/V_{\ell}$ in the mean-field model in the thermodynamic limit

were studied for the standard example, described in §3 of [2] in this volume, by Davies [4], Wreszinski [5], Fannes and Verbeure [6] and Buffet and Pule [7]. The mean-field model in the general situation, where the only assumptions about the single-particle spectrum are that (S1) and (S2) hold, was investigated in [8]; we have summarized the results of [8] in §2 and now go on to investigate the fluctuations in X_ℓ . In fact, we do rather more; we regard $\lambda \rightarrow X_\ell(\lambda)$ as a stochastic process and prove a central limit theorem:

THEOREM 4

Let $Z_\ell(\lambda) = V_\ell^{1/2} \{X_\ell(\lambda) - \tilde{\mathbb{E}}_\ell^\mu[X_\ell(\lambda)]\}$; then, for $\mu < a\rho_C$, $Z_\ell(\lambda) \xrightarrow{(d)} Z(\lambda)$, where $Z(\lambda)$ is gaussian with mean zero and covariance $\Gamma(\lambda_1, \lambda_2)$ given by

$$\Gamma(\lambda_1, \lambda_2) = J_{\lambda_1 \wedge \lambda_2}^\mu - \frac{aJ_{\lambda_1}^\mu J_{\lambda_2}^\mu}{1 + aJ_\infty^\mu}, \quad (3.1)$$

where

$$J_\lambda^\mu = \int_{[0, \lambda)} p''(\alpha(\mu) | \lambda) dF(\lambda), \quad (3.2)$$

and $\alpha(\mu)$ is the value of α at which $\inf_{\alpha < 0} \left\{ \frac{(\mu - \alpha)^2}{2a} + p(\alpha) \right\}$ is attained.

Sketch of proof:

The result follows from a routine, but somewhat tedious, calculation of

$$\lim_{\ell \rightarrow \infty} \tilde{\mathbb{E}}_\ell^\mu [e^{\beta(s_1 Z_\ell(\lambda_1) + s_2 Z_\ell(\lambda_2))}]$$

along the lines of the proof of Theorem 3. ■

It is interesting to identify the process $Z(\cdot)$ in terms of a standard process.

THEOREM 5

Let $B(\cdot)$ be a BM(1), a brownian motion in \mathbb{R}^1 starting at zero; then, for $\mu < a\rho_C$

$$Z(\lambda) \stackrel{(d)}{=} B(J_\lambda^\mu) - \frac{aJ_\lambda^\mu}{1 + aJ_\infty^\mu} B(J_\infty^\mu + 1/a). \quad (3.3)$$

Proof:

A routine computation shows that the mean of the right-hand side of (3.3) is zero and the covariance is the same as that of $Z(\cdot)$, given by (3.1). Hence the two gaussian processes are equal in distribution. ■

The process (3.3) is a modification of a time-changed brownian bridge; it never reaches the point at which it is tied-down but, as a increases, that point comes closer to J_{∞}^{μ} . This shows how, as the strength of the interaction increases, the fluctuations in $Z(\infty)$ are damped down.

It is a little more difficult to deal with the case $\mu > a\rho_c$; we introduce

$$W_{\mu}(\lambda) = Z_{\mu}(\infty) - Z_{\mu}(\lambda) \quad (3.4)$$

and prove in analagous fashion:

THEOREM 6

For $\mu > a\rho_c$, $W_{\mu}(\lambda) \xrightarrow{(d)} W(\lambda)$, where $W(\lambda)$ is a gaussian process with mean zero and covariance $\Gamma(\lambda_1, \lambda_2)$ given by

$$\Gamma(\lambda_1, \lambda_2) = K_{\lambda_1}^{\mu} \vee \lambda_2, \quad (3.5)$$

where

$$K_{\lambda}^{\mu} = \int_{[\lambda, \infty)} p''(0|\lambda) dF(\lambda). \quad (3.6)$$

In this case,

$$W(\lambda) \stackrel{(d)}{=} K_{\lambda}^{\mu} B\left[\frac{1}{K_{\lambda}^{\mu}}\right]. \quad (3.7)$$

The method by which we discovered the representations may be of some interest. The stochastic differential equation satisfied by a process $(X_t)_{t \geq 0}$ with filtration (\mathcal{F}_t) is discussed by Nelson [9]; see also McGill [10].

Suppose that a process (X_t, \mathcal{F}_t) satisfies the stochastic differential equation

$$X_t = X_s + \int_s^t \sigma(u, X_u) dB(u) + \int_s^t \tau(u, X_u) du; \quad (3.8)$$

then

$$\tau(s, X_s) = \lim_{t \downarrow s} \frac{1}{t-s} \mathbb{E} [X_t - X_s | \mathcal{F}_s] \quad (3.9)$$

and

$$\sigma^2(s, X_s) = \lim_{t \downarrow s} \frac{1}{t-s} \mathbb{E} [X_t - X_s]^2 | \mathcal{F}_s]. \quad (3.10)$$

Assuming that the processes $Z(\lambda), W(\lambda)$ satisfy stochastic differential equations, the corresponding coefficients σ and τ can be computed using (3.9) and (3.10); this is a routine exercise starting from the expressions (3.1) and (3.5) for the covariances since the processes are gaussian. Obvious time-changes then give the stochastic differential equations for a brownian bridge and a brownian motion respectively.

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