The Torricelli-Fermat Point Generalised

J. L. Synge

Dublin Institute for Advanced Studies

Abstract: The Torricelli-Fermat point (TF-point) of a triangle is that point which minimises the sum of its distances from the vertices. I generalise this definition, replacing the triangle by a set of M+1 points in E^N. Using the theory of convex functions, I show that the TF-point is unique and find explicit conditions to determine whether it coincides with any of the given points. If it does not, it may be found by solving a set of ordinary differential equations.

1. Introduction. In the geometry of the triangle there are certain familiar points - centroid, circumcentre, etc. The point discussed in this note is much less familiar: it is that point which minimises the sum of its distances from the vertices of a given triangle. It is strange that this point should be so little known: one can think of obvious applications, such as the location of a centre to supply three outposts with a minimum of distance travelled.

The problem is a very old one, having been stated by Fermat (1601-1665) and solved by Torricelli (1608-1647), but only for acute-angled triangles. Coxeter[1] describes a proof due to Hofmann in 1929 and remarks that the restriction to acute-angled triangles was removed by Pedoe in 1957. In correspondence with me Coxeter has suggested that the point should be called Torricelli-Fermat (briefly TF-point), and I adopt that name.

In the present paper I generalise the problem: To find the point P which minimises the sum of distances

$$S(P) = PA_o + PA_1 + \dots + PA_M, \qquad (1.1)$$

where the A's are given points in Euclidean N-space with $M \geqslant 2$, $N \geqslant 2$ and no three points are collinear.

For the classical problem $M = N = 2$ and the three points form an undegenerate triangle.

2. Notation. Vectors in E^N are indicated by heavy type. $\mathbf{A}_i$ ($i = 0,1,\dots,M$) are the position vectors of given points relative to an arbitrary origin O. Scalar products are indicated by dots. If $\mathbf{P}$ is the position vector of an arbitrary point, (1.1) may be written

$$S(\mathbf{P}) = \sum_{i=0}^{M} [(\mathbf{P}-\mathbf{A}_i) \cdot (\mathbf{P}-\mathbf{A}_i)]^{1/2}. \qquad (2.1)$$

If we give an arbitrary infinitesimal displacement to $\mathbf{P}$, we have

$$dS(\mathbf{P}) = -d\mathbf{P}.\mathbf{Q}, \qquad (2.2)$$

where $\mathbf{Q}$ is a sum of unit vectors

$$Q = \sum_{i=0}^{M} I_i, \qquad I_i = (A_i - P)/PA_i. \qquad (2.3)$$

These unit vectors are drawn from P in the directions of the A-points, and are well defined unless P coincides with an A-point, in which case the corresponding I-vector does not exist.

3. Theorem I: The TF-point exists and is unique.

Proof: Since $S(P)$ as in (2.1) is positive, there is at least one point P at which it has an absolute minimum. Thus at least one TF-point exists. To prove uniqueness, one appeals to the theory of convex functions.[2] A function f(x) is convex if it satisfies

$$f[\theta x_1 + (1-\theta)x_2] \leqslant \theta f(x_1) + (1-\theta)f(x_2) \qquad (3.1)$$

for every pair of distinct values of x_1, x_2 and for all θ in the open range (0,1). This means that the graph of f(x) from x_1 to x_2, excluding end-points, lies below or on (but not above) the straight line joining the end-points of the graph. For a strictly convex function the sign of equality in (3.1) is deleted; the graph of f(x) lies below the straight line joining the end points of the graph.

It is easy to see the sum of convex functions is itself convex, and a set of functions of which some are convex and some strictly convex is itself strictly convex.

Suppose now that there are two TF-points. Let L be the infinite straight line through them and x a measure of length on it, so that, if P lies on L, we may write $S(P) = \sum_{i=0}^{M} f_i(x)$. If A_i is not on L, a simple calculation shows that $f_i''(x)$ is positive, and this implies strict convexity. If A_i lies on L it is easy to see that $f_i(x)$ is convex. Since we have assumed that no three A-points are collinear, the sum S(P) contains at least one strictly convex function, and so S(P) on L is a strictly convex function of x, and it is known that a strictly convex function has at most one minimum. Thus the assumption of two TF-points is false, and uniqueness is proved.

<u>Theorem II</u>: If S(P) has a local minimum or a stationary value for some point P, then P is the TF-point.

<u>Proof</u>: Let T be the TF-point. Suppose that P is not T. Draw an infinite straight line L through P and T, with x a measure of distance on L. Then S = f(x) on L, and this function is strictly convex; this is inconsistent with the assumption that P is not T. Therefore P is T, and the theorem is proved.

<u>Theorem III</u>: A point P which is not one of the given points is the TF-point iff

$$\underset{\sim}{Q} = \underset{\sim}{I}_0 + \underset{\sim}{I}_1 + \dots + \underset{\sim}{I}_M = 0, \qquad (3.2)$$

where these are the unit vectors drawn from P towards the A-points, that is

$$\underset{\sim}{I}_i = (\underset{\sim}{A}_i - \underset{\sim}{P})/A_iP \quad . \qquad (3.3)$$

Proof: This follows immediately from Theorem II, the variation dS being given by (2.2).

4. Theorem IV: The TF-point is at A_o iff

$$\sum \cos \phi_{ij} \leq (1-M)/2 \ , \qquad (4.1)$$

where i and j run 1 to M with j < i and ϕ_{ij} is the angle between the vectors $\underset{\sim}{A}_i - \underset{\sim}{A}_o$ and $\underset{\sim}{A}_j - \underset{\sim}{A}_o$.

Proof: Take the origin at A_o. The position vector of any point P may then be written $s\underset{\sim}{I}$ where $\underset{\sim}{I}$ is a unit vector and s is the distance PA_o. Giving all directions to $\underset{\sim}{I}$ and letting s take all positive values, we cover the whole of E^N except the origin where s = 0. Then the sum S as in (2.1) is

$$S(P) = s + \sum_{i=1}^{M} [(s\underset{\sim}{I} - \underset{\sim}{A}_i) \cdot (s\underset{\sim}{I} - \underset{\sim}{A}_i)]^{1/2} \qquad (4.2)$$

Differentiating with respect to s and letting s tend to zero, we get

$$(dS/ds)_o = 1 - \underset{\sim}{I} . \underset{\sim}{R} \qquad (4.3)$$

where

$$\underset{\sim}{R} = \underset{\sim}{I}_1 + \underset{\sim}{I}_2 + \ldots + \underset{\sim}{I}_M, \qquad \underset{\sim}{I}_i = \underset{\sim}{A}_i / (\underset{\sim}{A}_i . \underset{\sim}{A}_i)^{1/2} \qquad (4.4)$$

these $\underset{\sim}{I}$'s being unit vectors drawn from A_o towards the other A-points.

Rotating the unit vector $\underset{\sim}{I}$ in all directions, the expression (4.3) is always positive iff the magnitude of $\underset{\sim}{R}$ is less than unity or equivalently

$$\underset{\sim}{R}.\underset{\sim}{R} < 1. \qquad (4.5)$$

But

$$\underset{\sim}{R}.\underset{\sim}{R} = M + 2\sum \cos \phi_{ij}, \qquad (4.6)$$

where the summation and the angles ϕ_{ij} are as in (4.1). Thus we have a local minimum, the equality sign following by continuity. This completes the proof.

In the classical case of a triangle, we have M = N = 2. Then the formula (4.1) tells us that the TF-point is at a vertex iff $\cos \phi \leqslant -1/2$, i.e. $\phi \geqslant 120^o$. For a tetrahedron in E^3, we have M = N = 3 and the vertex A_o is the TF-point iff

$$\cos \phi_{o1} + \cos \phi_{o2} + \cos \phi_{o3} \leqslant -1, \qquad (4.7)$$

these being the angles at A_o of the faces containing A_o.

5. <u>The TF-congruence.</u> Given the points A_i (i=0,1,...M) in E^N and seeking the TF-point, the systematic plan is first to test whether it lies at one of the A-points. This is done by investigating the inequality (4.1).

Suppose that the result is negative: then we must seek the TF-point elsewhere.

By Theorem II we know that we need only apply a stationary condition. Now by (2.2)

$$dS(\mathbf{P}) = -\,d\mathbf{P}.\mathbf{Q}, \qquad \mathbf{Q} = \sum_{i=0}^{M} \mathbf{I}_i, \qquad \mathbf{I}_i = (\mathbf{A}_i - \mathbf{P})/PA_i. \quad (5.1)$$

The stationary points are such that $\mathbf{Q}=0$. That condition is not easy to apply, but if we choose

$$d\mathbf{P} = \mathbf{Q}.ds \qquad (5.2)$$

where ds is an element of distance, we have

$$dS(\mathbf{P})/ds = -\mathbf{Q}.\mathbf{Q} \ . \qquad (5.3)$$

This differeential equation defines a congruence of curves in E^N, and if we proceed in the correct sense along any one of these curves, $S(\mathbf{P})$ steadily decreases. Since we have ruled out the A-points as possible TF-points, this congruence of curves must lead us to the TF-point, no matter where we start. Note that

$$\mathbf{Q}.\mathbf{Q} = M + 1 + \sum \cos \phi_{ij}, \qquad (5.4)$$

where in the summation $i = 0,1,\ldots M$ and $j < 1$.

6. <u>The tetrahedron</u>. The tetrahedron in E^3 stands next in simplicity to the triangle. In (4.1) we have the

conditions that the TF-point should be at a vertex. If it is not there, it is to satisfy (3.2), which it is convenient to write

$$\mathbf{Q} = \mathbf{I} + \mathbf{J} + \mathbf{K} + \mathbf{L} = \mathbf{0}, \qquad (6.1)$$

where these are unit vectors drawn from the TF-point towards the vertices A, B, C, D.

If we transfer L to the other side and square, we get

$$\mathbf{J}.\mathbf{K} + \mathbf{K}.\mathbf{I} + \mathbf{I}.\mathbf{J} = -1, \qquad (6.2)$$

a result of apparently little interest. But if we transfer both $\mathbf{K}$ and $\mathbf{L}$ to the other side and square, we get

$$\mathbf{I}.\mathbf{J} = \mathbf{K}.\mathbf{L}. \qquad (6.3)$$

Thus at the TF-point the sides AB and CD subtend the same angle. Obviously this is true for all the three pairs of opposite sides of the tetrahedron.

This suggests a construction for the TF-point. With AB as chord, describe a circular arc containing an angle θ and rotate this arc around AB, forming a spindle. If θ changes continuously from π to zero, the growing spindle covers all space. If we do the same with CD, using an angle ϕ, we shall get a second system of spindles. But if we make $\phi = \theta$ and let their common ~~angle~~ value decrease from π, there will be a state in which the two spindles touch, and this will be the TF-point of the tetrahedron. Since this point is unique, we see that there is a unique point (the TF-point) at which ~~in~~ for each pair of opposite edges, the two edges subtend the same angle.

7. <u>Conclusion</u>. I thank my colleague Professor J. T. Lewis for discussions, and in particular for suggesting the use of convexity to establish uniqueness. I also thank Professor H. S. M. Coxeter for correspondence.

<u>References</u>.

1. H. S. M. Coxeter, Introduction to Geometry, Wiley 1961, p. 21.

2. R. T. Rockafellar, Convex analysis, Princeton Univ. Press 1972.