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# Supersymmetry in a BCS-Umklapp Model

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Abstract

We consider an extension of the BCS model which includes umklapp processes, and give a condition such that this model be supersymmetric within an SU(2|2) algebra. We show that a mean field fermion reduction of the model is diagonalizable provided the same condition is satisfied.

# Supersymmetry in a BCS-Umklapp Model

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## ABSTRACT

*We consider an extension of the BCS model which includes umklapp processes, and give a condition such that this model be supersymmetric within an* su(2|2) *algebra. We show that a mean field* fermion *reduction of the model is diagonalizable provided the same condition is satisfied.*

A standard Lie algebraic approach [1] to a hamiltonian $H$ of an interacting fermion system, where

$$H = \sum_i \varepsilon_i {a_i}^\dagger a_i + \frac{1}{2} \sum_{i,j,l,k} < i\ j \mid V \mid k\ l > {a_i}^\dagger {a_j}^\dagger a_l a_k, \tag{1}$$

with

$$\{a_k, a_{k'}\} = 0 \ \ ; \ \ \{a_k, {a_{k'}}^\dagger\} = \delta_{k,k'} \ \ ; \ \ k \equiv (\mathbf{k}, \Uparrow), \ -k \equiv (-\mathbf{k}, \Downarrow), \tag{2}$$

proceeds as follows.

*i*) By means of some *linearization* procedure, one reduces $H$ to

$$H^{red} = \sum_i \varepsilon_i {a_i}^\dagger a_i + \sum \left( pairs\ of\ a's \right), \tag{3}$$

which is now an element of a Lie algebra $\mathfrak{L}$.

*ii*) The spectrum is obtained by means of a generalized *Bogolubov transformation* which is an automorphism $\Phi$: $\mathfrak{L} \to \mathfrak{L}$ such that

$$\Phi(H^{red}) = \alpha_1 h_1 + \ldots + \alpha_l h_l \ \ , \tag{4}$$

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where the set $\{h_1,\dots,h_l;e_1,\dots,e_{n-l}\}$ is a Cartan basis for the $n$-dimensional rank-$l$ Lie algebra $\mathfrak{L}$.

*iii*) The Cartan elements $\{h_1,\dots,h_l\}$ represent *observables which are conserved* in the high temperature phase, but no longer conserved in some low temperature phase.

*iv*) The remaining basis elements $\{e_1,\dots,e_{n-l}\}$ represent *order operators* whose expectations $< e_i >$ give the relevant order parameters.

*v*) *Coherent states* [2] are obtained by the action of a unitary operator $U$ which implements the automorphism $\Phi$; e.g. the coherent state given by $\mid \Omega >= U^{-1} \mid \omega >$ corresponds to the cyclic vector $\mid \omega >$ which is the vacuum for the diagonalized $H^{red}$.

We can implement the linearization procedure *i*) as follows. We consider the identity

$$AB = (A-<A>)(B-<B>)+<A>B+A<B>-<A><B>, \tag{5}$$

where $< \bullet >$ is the expectation in some state. If the first term at the r.h.s. can be considered "small" in some sense, this linearizes to

$$AB \approx <A>B+A<B>-<A><B>. \tag{6}$$

This approximation is consistent only in the following cases:

*a*) $[A,B]=0$; $A$ and $B$ are *bosonic*. This is the case, for example, of the standard mean field reduction of hamiltonian (1), where $A=a_i{}^\dagger a_{-i}{}^\dagger$, $A=a_{-j}a_j$.

*b*) $\{A,B\}=0$; $A$ and $B$ are *fermionic*. Then $AB=-BA$ requires that $\vartheta_A =< A >$ and $\vartheta_B =< B >$ be anticommuting numbers which anticommute as well with the operators $A$ and $B$.

We exemplify this procedure by a generalization of the BCS model which includes umklapp processes.

From the interaction part of the hamiltonian (1) we retain only the following terms

1) *Cooper-pairing* terms (BCS): $\frac{1}{2}\sum_{i,j} < i\ -i \mid V \mid j\ -j > a_i{}^\dagger a_{-i}{}^\dagger a_{-j} a_j$ .

2) *Umklapp* terms (U): $\frac{1}{2}\sum'_{i,j} < i\, j \mid V \mid -j\ -i > a_i{}^\dagger a_j{}^\dagger a_{-i} a_{-j}$. These terms are permitted in a crystal where momentum needs only be conserved modulo a wave vector of the reciprocal lattice $\mathbf{L}$ (the prime indicates this restriction on the summation).

Using the linearization procedure of case $a$), our reduced hamiltonian is now of the form $H^{(1)} = \sum_i H_i^{(1)}$, where

$$H_k^{(1)} = \varepsilon_k(a_k{}^\dagger a_k + a_{-k}{}^\dagger a_{-k}) + (\Delta_k a_k{}^\dagger a_{-k}{}^\dagger + v_k a_k{}^\dagger a_{-k} + \text{h.c.}); \tag{7}$$

$$\Delta_k = \frac{1}{2} \sum_j < k \ -k \,|\, V \,|\, j \ -j >< a_j a_{-j} >; \tag{8}$$

$$v_k = \frac{1}{2} \sum_j < k \ j \,|\, V \,|\, -j \ -k >< a_j{}^\dagger a_{-j} > . \tag{9}$$

The dynamical Lie algebra for this BCS-U model is $\bigoplus_k (su(2) \oplus su(2))_k$ generated by

$$J_+{}^{(k)} = \left(J_-{}^{(k)}\right)^\dagger = a_k{}^\dagger a_{-k}{}^\dagger \ , \ J_3{}^{(k)} = \frac{1}{2}(a_k{}^\dagger a_k + a_{-k}{}^\dagger a_{-k} - 1);$$

$$\tilde{J}_+^{(k)} = \left(\tilde{J}_-^{(k)}\right)^\dagger = a_k{}^\dagger a_{-k} \ , \ \tilde{J}_3^{(k)} = \frac{1}{2}(a_k{}^\dagger a_k - a_{-k}{}^\dagger a_{-k}). \tag{10}$$

The spectrum is easily obtained by means of the Bogolubov transformation

$$H_k{}^{(1)} \longmapsto \sqrt{\varepsilon_k{}^2 + |\Delta_k|^2}\ (a_k{}^\dagger a_k + a_{-k}{}^\dagger a_{-k} - 1) + |v_k|(a_k{}^\dagger a_k - a_{-k}{}^\dagger a_{-k}), \tag{11}$$

and the coherent states follow as outlined above.

We now add fermionic operators to the BCS-U model, including the following additional umklapp terms,

3) $\frac{1}{2} \sum'_{i,k} < i \ -i \,|\, V \,|\, k \ i > a_i{}^\dagger a_{-i}{}^\dagger a_i a_k \ ; \ (\mathbf{i} + \mathbf{k}) \in \mathbf{L}$,

4) $\frac{1}{2} \sum'_{i,k} < i \ -i \,|\, V \,|\, k \ -i > a_i{}^\dagger a_{-i}{}^\dagger a_{-i} a_k \ ; \ (\mathbf{i} - \mathbf{k}) \in \mathbf{L}$.

We use the linearization procedure $b$) on these terms, so that, for example,

$$a_i{}^\dagger a_{-i}{}^\dagger a_i a_k \ \approx < a_i{}^\dagger a_{-i}{}^\dagger a_i > a_k + a_i{}^\dagger a_{-i}{}^\dagger a_i < a_k >$$

to obtain a new reduced hamiltonian $H^{(2)} = \sum_k H_k{}^{(2)}$ of the form

$$H_k{}^{(2)} = \sum_{i=1}^{6} b_i B_i + \sum_{j=0}^{8} f_j F_j \ \in \ su(2\,|\,2) \tag{12}$$

where we suppressed the $k$-dependence on the r.h.s.. The operators $B_i$ , $i = 1, \ldots, 6$ are the generators of the $(su(2) \oplus su(2))_k$ algebra introduced above in (10); while the $F_j$ , $j = 1, \ldots, 8$ are the fermionic operators

$$\{a_k, a_{-k}, a_k{}^\dagger, a_{-k}{}^\dagger, n_k a_{-k}, n_{-k} a_k, a_{-k}{}^\dagger n_k, a_k{}^\dagger n_{-k}\},$$

where $n_k \equiv {a_k}^\dagger a_k$. The set $\{B_1, \ldots, B_6; F_0, F_1, \ldots, F_8\}$ (where $F_0 \equiv \mathbf{I}$ was added) forms a basis for the superalgebra $su(2|2)_k$. The coefficients $b_i, f_i$ are elements of the extension ring $\mathbb{C}[\vartheta_\infty, \vartheta_\in, \ldots]$ generated by the $\vartheta$-terms, which are expectations of odd numbers of fermions arising from the linearization procedure $b$).

This model has been treated in ref.[3], where the finite- temperature self-consistency equations (which are independent of $\vartheta$) were written down.

Within the context of the $su(2|2)$ superalgebra, it was shown in ref.[3] that the hamiltonian $H^{(1)}$ is *supersymmetric*; that is we may define a charge $Q \in \mathfrak{F}(\bigoplus_k su(2|2)_k)$ ($\mathfrak{F}$ denoting the fermionic sector) such that

$$H^{(1)} = \{Q, Q^\dagger\} \quad , \quad Q^2 = 0 \quad , \quad [H^{(1)}, Q] = 0 \; . \tag{13}$$

This is only possible when the coefficients in (7) satisfy the following condition

$$|v_k|^2 = |\Delta_k|^2 + \varepsilon_k^2. \tag{14}$$

We now treat $H^{(1)}$ by means of a *self-consistent mean-field Fermi reduction* using the linearization process $b$) on the interaction terms. This produces the following hamiltonian

$$\begin{aligned} {H_k}^F = \varepsilon_k(n_k + n_{-k}) + \{\Delta_k(< {a_k}^\dagger > {a_{-k}}^\dagger + {a_k}^\dagger < {a_{-k}}^\dagger >) \\ + v_k(< {a_k}^\dagger > a_{-k} + {a_k}^\dagger < a_{-k} >) + \text{h.c.}\}. \end{aligned} \tag{15}$$

Define

$$\begin{aligned} {\vartheta_-}^{(0)}(k) &= -\overline{\Delta}_k < a_k > + v_k < {a_k}^\dagger > \; , \\ {\vartheta_+}^{(0)}(k) &= \overline{\Delta}_k < a_{-k} > + \overline{v}_k < {a_{-k}}^\dagger > \; , \end{aligned} \tag{16}$$

and write

$$a(\vartheta_\pm(k)) \equiv \vartheta_\pm(k) a_{\pm k} \quad ; \quad a^\dagger(\overline{\vartheta}_\pm(k)) \equiv \; {a_{\pm k}}^\dagger \overline{\vartheta}_\pm(k) = [a(\vartheta_\pm(k))]^\dagger . \tag{17}$$

With this notation the hamiltonian ${H_k}^F$ becomes

$${H_k}^F = \varepsilon_k(n_k + n_{-k}) + \{a({\vartheta_-}^{(0)}(k)) + a({\vartheta_+}^{(0)}(k)) + \text{h.c.}\} \; , \tag{18}$$

which is an element of a solvable SLA $\mathcal{A}_k \subset su(2|2)_k$.

To diagonalize $H^F$, we consider the adjoint action $\exp(\text{ad } iZ)$ of an element $Z \in \mathcal{A}$, where $\mathcal{A} = \bigoplus_k \mathcal{A}_k$ , $Z = \bigoplus_k Z_k$ , and

$$Z_k = \{a(\vartheta_+(k)) + a(\vartheta_-(k)) + \text{h.c.}\}. \tag{19}$$

The condition that $\exp(\text{ad } iZ)(H^F) \equiv \mathcal{U}(\vartheta) H^F \mathcal{U}^{-1}(\vartheta)$ be free of non-diagonal terms is

$$\vartheta_\pm(k) = \frac{i}{\varepsilon_k} \vartheta_\pm{}^{(0)}(k). \tag{20}$$

We may evaluate the expectation of any operator $\mathcal{O}$ in the *supercoherent state* $|\tilde{\Omega}> = \mathcal{U}^{-1}(\vartheta)|\tilde{\omega}>$ by

$$\begin{aligned} <\tilde{\Omega}|\mathcal{O}|\tilde{\Omega}> &= <\tilde{\omega}|\mathcal{U}(\vartheta)\mathcal{O}\mathcal{U}^{-1}(\vartheta)|\tilde{\omega}> \\ &= <\tilde{\omega}|\exp(i \text{ ad} Z)(\mathcal{O})|\tilde{\omega}> . \end{aligned} \tag{21}$$

In particular, for the single-fermion operator expectation we have

$$< a(\vartheta_+(k)) > = i\bar{\vartheta}_+(k)\vartheta_+(k) \quad ,i.e. \quad < a_k > = -i\bar{\vartheta}_+(k); \tag{22}$$

thus, using eq.(20), $< a_k >= -\bar{\vartheta}_+{}^{(0)}(k)/\varepsilon_k$.

However, by definition (16),

$$< a_k > = \frac{\Delta_k \vartheta_-{}^{(0)}(k) + v_k \bar{\vartheta}_-{}^{(0)}(k)}{|v_k|^2 - |\Delta_k|^2}.$$

We have similar equations for $< a_{-k} >$ , $< a_k{}^\dagger >$ , $< a_{-k}{}^\dagger >$. We thus obtain four linear equations homogeneous in $\vartheta_+{}^{(0)}(k)$ , $\vartheta_-{}^{(0)}(k)$ , $\bar{\vartheta}_+{}^{(0)}(k)$ , $\bar{\vartheta}_-{}^{(0)}(k)$ , leading to the determinantal condition

$$|v_k|^2 = |\Delta_k|^2 + \varepsilon_k^2, \tag{23}$$

which is the same as eq.(14) for the hamiltonian $H^{(1)}$ to be supersymmetric.

The superalgebraic approach outlined in this note may be generalized to more complex interacting fermion systems. In an $n$-fermion problem defined by anticommuting operators $\{a_1, \ldots, a_n; a_1^\dagger, \ldots, a_n^\dagger\}$ the superalgebra generated by all possible combinations is $su(2^{n-1}|2^{n-1})$ of dimension $2^{2n} - 1$ . For example,

| | | | | |
|---|---|---|---|---|
| $n=2$ | BCS type (singlet) models | $\in$ | $su(2\|2)$ | $dim=15$ |
| $n=4$ | Helium-3 type (triplet) models | $\in$ | $su(8\|8)$ | $dim=255$ |
| $n=8$ | Superconducting density wave models | $\in$ | $su(128\|128)$ | $dim=65535$ |

Purely Lie-algebraic treatments of the $n = 4$ and $n = 8$ cases are given in refs.[1] and [4] respectively. The rapid growth of the dimension in the superalgebraic case indicates an increasing complexity of structure; some analysis of the $n = 4$ case has already been made [5].


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