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The Anomaly-Flux-Index Identity and its Euclidean Extension

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ABSTRACT

The identity of the $U(1)$ anomaly A, the magnetic flux Φ and the Atiyah-Singer index I ($A = \Phi = I$) for 2n-dimensional compact manifolds is recalled and established in a simple manner by identifying each of them with the central quantity $Q = m^2 \; tr \; \gamma(\not{D}^2 + m^2)^{-1}$, where γ is the 2n-dimensional analogue of γ_5, and it is shown that for Euclidean manifolds the identity holds if the index $I = (n_+ - n_-)$ is replaced by the quantity $\tilde{I} = (n_+ - n_-) + \frac{1}{\pi}(\eta_+(0) - \eta_-(0))$ where $\eta_\pm(0)$ are sums over zero-energy phase-shifts.

1. INTRODUCTION.

In recent years the so-called anomalies of gauge-field theories (and their absences) have come to play an increasingly important role. For example it is the requirement that the $U(1)$ anomalies of the standard electroweak-theory be absent that leads to equal numbers of quark and lepton generations[1] and it was the discovery that superstring theory was anomaly-free that started the present

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wave of interest in that theory[2]. Another interesting aspect of anomalies, however, is that they establish links between hitherto unconnected pieces of physics and mathematics. For compact manifolds, for example, it is now realized that the $U(1)$-anomaly A of Adler et al[3], the magnetic flux Φ and the Atiyah-Singer index I are identical ($A = \Phi = I$) and the first part of this lecture will be devoted to showing how this identity can easily be established by relating all three quantities to a central quantity (see Fig. 1) Q defined as

$$Q = tr\left[\gamma\left(\frac{m^2}{m^2 + \not{D}^2}\right)\right], \quad \not{D} = \not{\partial} + e\not{A}, \tag{1.1}$$

where $\not{D}$ is the Dirac operator in 2n-dimensions, and $\gamma = \pm 1$ the generalization of the Dirac γ_5 that distinguishes between the two spinor representations of SO(2n).

For non-compact manifolds the identity $Q = A = \Phi$ continues to hold (indeed the proof of these relations makes no use of the compactness of the manifolds i.e. the discreteness of the spectrum of $\not{D}$) but the identify of the other quantities with I cannot continue to hold since I is an integer but the others are not necessarily so. However, the index can be replaced by a modified non-integer quantity $\tilde{I}$ for which the identity does hold and one such modification, for which the fractional part of I is called the η-invariant[5] and which is defined on non-compact manifolds with boundaries, has been much discussed in the recent literature[6]. Here we wish to discuss a modification which is defined on the Euclidean manifold of the original anomaly[3] and for which the fractional part turns out to be the phase-shift for low-energy scattering[7]. In fact, instead of $A = \Phi = I \equiv (n_+ - n_-)$ one obtains in the Euclidean case the identity

$$A = \Phi = \tilde{I} = (n_+ - n_-) + \frac{1}{\pi}\sum_{\ell}\left(\eta_\ell^+(0) - \eta_\ell^-(0)\right) \tag{1.2}$$

where $\pm$ denotes chirality and $\eta_\ell(0)$ the scattering phase-shifts in the limit of zero energy. The great advantage of this Euclidean version of the index theorem is that it refers directly to physical quantities, and that it relates two further pieces of physics that were hitherto unrelated, namely, the Levinson theorem (obtained from (1.2) for $\Phi = 0$) and the Bohm-Aharonov effect (obtained from (1.2) for $n_+ = n_-$).

2. IDENTIFICATION OF THE ANOMALY WITH Q.

We begin with the relation $Q = A$ (anomaly), although this is in some sense the least satisfactory of the three relations ($Q = A = \Phi = I$) because the anomaly A needs field theory and ultra-violet regularization to place it in context[1][8]. The starting point is the Schwinger functional for the Dirac operator (at zero external current), namely,

$$\begin{aligned} J &= \ln \int d(\psi\bar{\psi}) \exp(\not{D} + iM) \\ &= \ln \det(\not{D} + iM) = \ tr\ \ln(\not{D} + iM), \quad M = m + i\gamma\mu, \end{aligned} \tag{2.1}$$

where the right hand side is assumed to be regularized (e.g. by subtracting a Pauli-Villars term) to ensure the ultra-violet convergence of the trace. The mass-term M has been inserted to ensure the infra-red existence of the logarithm, the form $M = m + i\gamma\mu$ being chosen to keep track of chiral variations, for which (m, μ) is supposed to be a doublet.

Then for (global) chiral transformations with (constant) parameter α one has

$$\not{D} \rightarrow e^{i\gamma\alpha}\not{D}e^{i\gamma\alpha} = \not{D} \quad \text{and} \quad M - M(\alpha) \ = e^{i\gamma}Me^{i\gamma} = Me^{2i\alpha\gamma}, \tag{2.2}$$

and hence

$$J \rightarrow J(\alpha) = \ tr\ \ln(\not{D} + iM(\alpha)). \tag{2.3}$$

From (2.3) one sees at once that

$$\frac{\partial J(\alpha)}{\partial \alpha} = -2\ tr\ \left[\left(\frac{M(\alpha)}{\not{D} + iM(\alpha)}\right)\gamma\right], \tag{2.4}$$

and an interesting feature of (2.4) is that the trace on the right-hand-side exists without any ultra-violet regularization. Thus while $J(\alpha)$ requires ultra-violet regularization $J'(\alpha)$ does not.

If now, following convention, one defines the global anomaly A as the chiral variation of $J(\alpha)$ at $\alpha = 0$ (with $\mu = 0$ and a factor $(2i)^{-1}$) one sees from (2.4) that

$$A = i\ tr\ \left[\left(\frac{m}{\not{D} + im}\right)\gamma\right]. \tag{2.5}$$

Hence, on multiplying above and below by $(\not{D} - im)$ and using the Dirac trace, one has

$$A = i\ tr\ \left[\left(\frac{m(\not{D} - im)}{(\not{D}^2 + m^2)}\right)\gamma\right] = tr\ \left[\left(\frac{m^2}{\not{D}^2 + m^2}\right)\gamma\right] = Q, \tag{2.6}$$

as required.

3. IDENTIFICATION OF THE FLUX WITH Q.

The magnetic flux Φ is defined (for the $U(1)$ case) to be

$$\Phi = \int d^{2n}\epsilon_{\alpha\beta\mu\nu...\lambda\sigma}F_{\alpha\beta}F_{\mu\nu}...F_{\lambda\sigma}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{3.1}$$

where $F_{\mu\nu}$ is the electromagnetic field in 2n-dimensions, and for $n = 1, 2$ Φ reduces to the familiar forms

$$\Phi_2 = \int d^2x B(x) \quad \text{and} \quad \Phi_4 = \int d^4x F_{\mu\nu}(x)F^*_{\mu\nu}(x), \tag{3.2}$$

respectively, where B is the 2-dimensional magnetic field, and star denotes dual. The identity $\Phi = Q$ is established in two steps. First $Q(e)$ is shown to be a polynomial of degree at most $2n$ in the coupling constant e and then the (consequently terminating) perturbation expansion is used to show that all coefficients in the polynomial vanish except the coefficient of e^n, which (for $e = 1$) is just Φ.

For the first step the idea is to note that the traces

$$g_p(e) = tr\ \left(\frac{1}{\not{D} + iM}\not{A}\right)^p, \tag{3.3}$$

exist in 2n-dimensions provided that $p > 2n + 1$. Furthermore, since $\bar{\psi}(\not{D} + im)\psi$ is invariant with respect to chiral transformations

$$\psi \to e^{i\gamma\alpha}\psi, \quad M \to Me^{2i\gamma\alpha}, \tag{3.4}$$

and the fermions ψ drop out in the (finite) trace, the $g_p(e)$ can be functions of m and μ only through the chiral-invariant combination $\sigma = m^2 + \mu^2$. On the other hand, one sees by inspection that

$$(\partial_\mu g_p(e))_{\mu=0} = p\, tr \left[\left(\frac{1}{\not{D}+iM}\right)\gamma\left(\frac{1}{\not{D}+iM}\not{A}\right)^p\right] \\ = \frac{1}{(p-1)!}\left(\frac{\partial}{\partial e}\right)^p tr\left[\gamma\left(\frac{1}{\not{D}+iM}\right)\right]. \tag{3.5}$$

But since, from (2.5), the trace in the last expression in (3.5) is just $Q(e)$, one then has

$$\left(\frac{\partial}{\partial e}\right)^p Q(e) = (p-1)!\left(\frac{\partial}{\partial\mu}g_p(e)\right)_{\mu=0} \\ = (p-1)!\left(2\mu\frac{\partial}{\partial\sigma}g_p(e)\right)_{\mu=0} = 0, \quad \text{for} \quad p > (2n+1), \tag{3.6}$$

and this shows that $Q(e)$ is a polynomial of degree at most $(2n+1)$, as required.

The second step is to use the perturbation expansion for $Q(e)$, which now terminates, and may be written in the form

$$Q(e) = tr\left[\gamma\left(\frac{1}{\not{D}+iM}\right)\right] = \sum_{s=0}^{2n+1} tr\left[\gamma\left(\frac{1}{\not{\partial}+im}\right)\left(\not{A}\frac{1}{\not{\partial}+im}\right)^s\right]. \tag{3.7}$$

The argument used above for $g_p(e)$, $p > 2n+1$ fails for $p \le 2n+1$ since the corresponding traces do not exist (otherwise $Q(e)$ would be zero and there would be no anomaly!). However, the denominators in (3.7) are the free-field denominators so ordinary Feynman graph techniques can be used. These techniques are too well-known to be worth reproducing here so I shall just indicate how the computation goes for the 2-dimensional case. In that case the series (3.7) has just three terms,

$$tr\left\{\left(\frac{m}{\not{\partial}+im}\right)\gamma\right\} + tr\left\{\left(\frac{m}{\not{\partial}+im}\not{A}\frac{m}{\not{\partial}+im}\right)\gamma\right\} \\ + tr\left\{\left(\frac{m}{\not{\partial}+im}\not{A}\frac{m}{\not{\partial}+im}\not{A}\frac{m}{\not{\partial}+im}\right)\gamma\right\}, \tag{3.8}$$

of which the first vanishes because the Dirac trace is zero and the third vanishes because it is a pseudo-scalar, but Lorentz and ordinary gauge invariance require it to be of the form $\int B_\mu(x)F((x-y)^2)B_\mu(x)d^2x$, which is a scalar. That leaves only the central term, which may be written as

$$tr\left\{\frac{m^2}{\partial^2+m^2}B(x)\frac{1}{(\partial^2+m^2)}\right\} = \int B(x)d^2x = \Phi, \tag{3.9}$$

as required.

4. IDENTIFICATION OF THE INDEX WITH Q (COMPACT CASE).

To identify the index with Q one notes first of all that since the Dirac operator $\not{D}$ anti-commutes with γ, it is completely off-diagonal when γ is diagonal,

$$\not{D} = \begin{pmatrix} O & D_+ \\ D_- & O \end{pmatrix} \quad \text{when} \quad \gamma = \begin{pmatrix} 1 & O \\ O & -1 \end{pmatrix}. \tag{4.1}$$

Hence in this basis Q takes the form

$$Q = tr\left(\frac{m^2}{D_+D_-+m^2} - \frac{m^2}{D_-D_++m^2}\right), \tag{4.2}$$

where tr now denotes the n-dimensional, rather than the 2n-dimensional, trace. If the manifold is compact, the spectra of $D_\pm D_\mp$ are discrete, and since

$$D_\pm D_\mp\psi = \lambda\psi \Rightarrow D_\mp D_\pm(\phi) = \lambda\phi, \quad \text{where} \quad \phi = D_\mp\psi, \tag{4.3}$$

one sees that the eigenvalues of $D_\pm D_\mp$ are the same (and have the same multiplicity), except possibly for the zero eigenvalues. From this observation and (4.2) it follows at once that

$$Q = (n_+ - n_-), \tag{4.4}$$

where $n_\pm$ are the multiplicities of the zero eigenvalues of $D_\pm D_\mp$ (equivalently $D_\mp$). Since $(n_+ - n_-)$ is just the index I this establishes the result. Note that what one has actually used in (4.2)(4.3) and (4.4) is that $D_\pm D_\mp$ are the two pieces of a supersymmetric quantum-mechanical Hamiltonian[9], namely,

$$H = D^2 + \sigma^\pm \cdot F, \quad \sigma^\pm_{\mu\nu} = \frac{1}{2}(1\pm\gamma)\sigma_{\mu\nu} \tag{4.5}$$

where $\sigma^\pm_{\mu\nu}$ are the generators of the spinor representations of $SO(2n)$.

5. IDENTIFICATION OF THE MODIFIED INDEX WITH Q (EUCLIDEAN CASE WITH m → O).

Since the identification $Q = I$ in the compact case relied heavily on the discreteness of the operators $D_\pm D_\mp$ it is clear that some new idea is required for the Euclidean case. However (4.2) is still valid in that case, and may be written in the form

$$Q = \sum_\ell \int d\epsilon \left(\frac{m^2}{m^2+\epsilon}\right) d\left(E_\ell^+(\epsilon) - E_\ell^-(\epsilon)\right) \ , \ D_\pm D_\mp = \int \epsilon dE^\pm(\epsilon), \qquad (5.1)$$

where $E_\ell^\pm(\epsilon)$ are the spectral measures for the Hamiltonians (4.5), at fixed ℓ, and ℓ denotes angular momenta quantum numbers in the spherically symmetric case, and more generally the discrete eigenvalues of complete set of operators that commute with $D_\pm D_\mp$. The problem is to compute $E_\ell^\pm(\epsilon)$ and the new idea is to use an old formula of quantum mechanics[10] which expresses the spectral measure of any Hamiltonian (whose potential vanishes at spatial infinity) in terms of the phase-shift. The formula is

$$d(E_\ell(\epsilon) - \overset{o}{E}_\ell(\epsilon)) = d(\eta_\ell(\epsilon)\chi(\epsilon)) \ , \ H_\ell = \int \epsilon dE_\ell(\epsilon), \qquad (5.2)$$

where $\overset{o}{E}_\ell(\epsilon)$ is the corresponding measure for the free Hamiltonian, $\eta_\ell(\epsilon)$ is the phase-shift for fixed ℓ and $\chi(\epsilon)$ is the characteristic function $\chi(\epsilon) = 0, 1$ for $\epsilon < 0\, \epsilon \geq 0$. This formula will be established in the appendix, and anticipating its establishment we insert (5.2) into (5.1) to obtain

$$Q = \frac{1}{\pi}\sum_\ell \left(\eta_\ell^+(0) - \eta_\ell^-(0)\right) + \frac{1}{\pi}\int_0^\infty \left(\frac{m^2}{m^2+\epsilon}\right)\eta_\ell'(\epsilon)d\epsilon. \qquad (5.3)$$

Since $\eta_\ell'(\epsilon)$ is a continuous function of ϵ for $\epsilon \geq 0$ the integral in (5.3) vanishes (like $m^2 \ln m$) as $m \to 0$. Hence in this limit we have

$$Q = \frac{1}{\pi}\sum_\ell \left(\eta_\ell^+(0) - \eta_\ell^-(0)\right) = (n_+ - n_-) + \frac{1}{\pi}\sum \left(\tilde{\eta}_\ell^+(0) - \tilde{\eta}_\ell^-(0)\right) \qquad (5.4)$$

where in the second equation $\tilde{\eta}$ denotes the proper-fractional part of η and we have used Levinson's theorem. Eq. (5.4) is the required modification of the index theorem for the Euclidean case. It is, perhaps, surprising that the step from (5.3) to (5.4) required the limit $m \to 0$, since Q is dimensionless and should therefore not depend on m, as was found explicitly for the compact case, and in the next section we shall show that at least in two dimensions Q is indeed independent of m so the limit $m \to 0$ is not actually necessary.

6. IDENTIFICATION OF THE MODIFIED INDEX WITH Q (EUCLIDEAN CASE FOR ANY m).

In this section we wish to show that, in two dimensions at least, equation (5.4) holds for any value of m^2 so that the limit $m \to 0$ is unnecessary. We believe that the same is true for higher dimensions, but it will be seen that the proof we give does not immediately generalize.

The natural gauge for this problem is the radial gauge $A_r = 0$ and in this gauge the Dirac equation for scattering becomes

$$\begin{pmatrix} 0 & D_+ \\ D_- & 0 \end{pmatrix} \begin{pmatrix} f_+ \\ f_- \end{pmatrix} = i\epsilon \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \quad \text{where } D_\pm = \pm e^{\pm i\phi}(\partial_r \pm \frac{1}{i} D_\phi). \qquad (A.1)$$

In the asymptotic ($r \to \infty$) region $A_\phi(r,\phi) \to a(\phi)/r$ and $a(\phi)$ can be gauged to ω where ω is a constant. In that case the asymptotic version of $(A.1)$ can be decomposed into eigenstates of the angular momentum operator $\frac{1}{i}\partial_\phi$ and takes the form

$$\begin{bmatrix} 0 & e^{i\phi}(\partial_r + \frac{\ell+\omega}{r}) \\ e^{-i\phi}(-\partial_r + \frac{\ell+\omega+1}{r}) & 0 \end{bmatrix} \begin{pmatrix} f_\ell^+ \\ f_\ell^- \end{pmatrix} = \epsilon \begin{pmatrix} f_\ell^+ \\ f_\ell^- \end{pmatrix}, \qquad (A.2)$$

for each eigenvalue ℓ. If a factor $\exp[-i\phi]$ is absorbed in f_ℓ^+ then $(A.2)$ becomes

$$\begin{bmatrix} 0 & (\partial_r + \frac{\ell+\omega}{r}) \\ (-\partial_r + \frac{\ell+\omega+1}{r}) & 0 \end{bmatrix} \begin{pmatrix} f_{\ell-1}^+ \\ f_\ell^- \end{pmatrix} = \epsilon \begin{pmatrix} f_{\ell-1}^+ \\ f_\ell^- \end{pmatrix}, \qquad (A.3)$$

the index on f_ℓ^+ changing by one unit because the factor $\exp(-i\phi)$ has angular momentum minus one. On squaring $(A.3)$ one obtains

$$\begin{bmatrix} -\partial_r^2 - \frac{1}{r}\partial_r + \frac{(\omega+\ell)^2}{r^2} & 0 \\ 0 & -\partial_r^2 - \frac{1}{r}\partial_r + \frac{(\omega+\ell+1)^2}{r^2} \end{bmatrix} \begin{pmatrix} f_{\ell-1}^+ \\ f_\ell^- \end{pmatrix} = \epsilon^2 \begin{pmatrix} f_{\ell-1}^+ \\ f_\ell^- \end{pmatrix}, \qquad (A.4)$$

which shows that the $f^{\pm}$ are actually Bessel functions. In fact, if one takes the correlation $(A.3)$ into account one sees that the solution of $(A.4)$ is

$$\begin{aligned} f^{+}_{\ell-1} &= \alpha J_{\omega+\ell}(\epsilon r) + \beta J_{-(\omega+\ell)}(\epsilon r) \\ f^{-}_{\ell} &= \alpha J_{\omega+\ell+1}(\epsilon r) - \beta J_{-(\omega+\ell+1)}(\epsilon r) \end{aligned} \tag{A.5}$$

with the <u>same</u> constants α, β in both cases. From the asymptotic properties of the Bessel functions ($J_\nu(x) \to x^{-1/2}\cos(x + \frac{\nu\pi}{2} - \frac{\pi}{4})$) one then sees that the phase shifts $\delta^{+}_{\ell-1}, \delta^{-}_{\ell}$ are

$$\frac{\pi}{4} + \tan^{-1}\left\{\frac{\alpha \mp \beta}{\alpha \pm \beta}\tan\left(\frac{\pi}{2}\omega_{\pm}\right)\right\}, \text{ where } \begin{aligned} \omega_{+} &= \omega + \ell \\ \omega_{-} &= \omega + \ell + 1 \end{aligned}, \tag{A.6}$$

and thus they are correlated as follows,

$$\delta^{+}_{\ell-1} - \delta^{-}_{\ell} = \frac{\pi}{2}\sigma \quad \text{where} \quad \sigma = sgn(\omega + \ell + 1)\ (= \pm 1). \tag{A.7}$$

Eqn.$(A.7)$ is evidently the continuum analogue of the supersymmetric relation (4.3) and, like (4.3), it leads to an infinite set of cancellations. In fact, from $(A.7)$ one sees that the sum over angular momenta (5.1) – (5.2) 'telescopes' into a difference of the two extreme angular momenta

$$\sum_{-M}^{N}(\delta^{+}_{\ell} - \delta^{-}_{\ell}) = \delta^{+}_{N} - \delta^{-}_{-M}. \tag{A.8}$$

There is not a complete cancellation because $(A.7)$ connects the phase-shifts only slantwise (see Fig. 2). Note that the slantwise action is due to the fact that the vertical (fixed ℓ) supersymmetry of the two-dimensional Dirac operator in $(A.2)$ becomes a slantwise supersymmetry for the radial Dirac operator in $(A.3)$.

Eq.$(A.8)$ shows that the contribution to the modified index actually comes only from the high angular momentum limit, and this essentially establishes the result because, for Bessel functions $J_\nu(x)$, large ν corresponds to small x, and hence to $\epsilon \to 0$, which is the limit obtained from $m \to 0$ in (5.3). However, some physical insight can be obtained by continuing the present line of argument and drawing the result directly from $(A.8)$. For this purpose, one recalls that angular momentum drives wave-functions away from the origin (with factors r^{ℓ}) and hence if one chooses the angular momenta M, N in $(A.8)$ so large they drive the wave function into the asymptotic region of the potential ($r > r_0$ say) then the solutions $(A.5)$ acquire the boundary conditions $J(\epsilon r_o) \simeq 0$. But in that case $\alpha \simeq 0$ and $\beta \simeq 0$ for δ^+ and δ^- respectively and the phase-shifts reduce to $\delta^+_N \sim \frac{\pi}{2}\omega$, $\delta^-_{-M} \sim -\frac{\pi}{2}\omega$. Thus as $M, N \to \infty$ $\delta^+_N - \delta^-_{-M} \to \pi\omega$ and since ω is just the flux one has $(\delta^+_N - \delta^-_{-M})/\pi \to$ flux, as required.

In higher dimensions the operator D_+ in $(A.1)$ generalizes to

$$D_+ = U(\Omega)\left(\delta_r + \frac{D_\Omega}{r}\right), \tag{A.9}$$

where a factor $r^{2(n-1)}$ has been taken out of the wave-functions, Ω denotes all polar angles, and $U(\Omega)$ is a unitary matrix. For example, in four dimensions one has

$$D_+ = D_t + i\vec{\sigma}\cdot\vec{D} \; = \; e^{i\vec{X}\cdot\vec{\sigma}}\left[\partial_r + \frac{i}{r}\left(\vec{\sigma}\cdot(\vec{K}+\vec{L})\phi_i\right)\frac{\partial}{\partial\phi_i}\right], \tag{A.10}$$

where $\phi_i \, (i = 1,2,3)$ are polar angles, r is the four dimensional length,

$$\frac{\vec{X}}{|X|} \; = \; \frac{\vec{x}}{|x|}, \; |X| \; = \tan^{-1}\frac{|x|}{r}, \quad \text{and} \quad K = \vec{x}\partial_t - t\vec{\partial}, \;\; \vec{L} = \vec{x}\times\vec{\partial}. \tag{A.11}$$

However, it is not so easy to proceed further because $U(\Omega)$ is no longer a step-operator for D_Ω and hence the 'slanted' supersymmetry, obtained on eliminating $U(\Omega)$ from $(A.9)$, is much more complicated in the higher dimensions.

It might be worth mentioning that the cancellation of phase-shifts in $(A.8)$, analogous to the cancellation of discrete energy-levels in (4.2), is not a characteristic of supersymmetry alone, but of supersymmetry and the scale covariance of the Dirac operator. In fact, supersymmetric potentials for which the phase-shift cancellation does not take place are known[12].

APPENDIX. THE SPECTRAL MEASURE OF HAMILTONIAN AND THE PHASE-SHIFT.

We wish to establish eq. (5.2) relating $E(\epsilon)$ and $\eta(\epsilon)$. For simplicity, and because the extension to gauge-potentials and to spherically asymmetric systems is not difficult, we shall consider only Hamiltonians of the conventional spherically symmetric form

$$H = \frac{1}{2}p^2 + V(r), \quad V(r) \to 0,\ r \to \infty, \quad H = \int \epsilon dE(\epsilon), \tag{A1}$$

for fixed angular momentum ℓ (whose index is suppressed). Let us then consider the energy-trace

$$T = tr\left(g(H) - g\,\overset{o}{H})\right) = \int \epsilon\, d\,(E(\epsilon) - \overset{o}{E}\,(\epsilon), \tag{A2}$$

where $\overset{o}{H}$ is the free-Hamiltonian and $g(H)$ any function of H for which this trace exists. The problem is that the spectrum of H is continuous, and to circumvent this we temporarily immerse the system in a sphere of radius R (in practice impose the boundary condition $\psi(R,\Omega) = 0$ on wave-functions $\psi(r,\Omega)$), where R is so large that $V(R) \approx 0$ and the continuum limit can be recovered for $R \to \infty$. For the immersed system (A2) becomes

$$T = \sum_s g(\epsilon_s) - g(\overset{o}{\epsilon}_s), \tag{A3}$$

where $\epsilon_s, \overset{o}{\epsilon}_s$ are the (now discrete) eigenvalues of $H, \overset{o}{H}$ and are assumed to correspond to each other in the sense that $\epsilon_n \to \overset{o}{\epsilon}_n$ as $V(r) \to 0$ (for all r). Now in the asymptotic region for V the wave-function takes the usual scattering form

$$\psi(x) \to (k_s r)^{\left(\frac{1-n}{2}\right)} \sin(k_s r + \eta_s), \quad \text{where} \quad \epsilon_s = \frac{1}{2}k_s^2. \tag{A4}$$

But because of the boundary condition $\psi(R) = 0$ the momenta $k_s\ \overset{o}{k}_s$ and the phase-shift η_s are related by the conditions

$$k_s R + \eta_s = s\pi \qquad \overset{o}{k}_s\, R = s\pi, \quad s\, integer \tag{A5}$$

(a result which was somewhat anticipated by using the subscript s for ϵ_s). On eliminating R and s from (A5) one obtains

$$\eta_s = -\pi\frac{(k_s - \overset{o}{k}_s)}{(\overset{o}{k}_{s+1} - \overset{o}{k}_s)} = -\pi\frac{\delta k_s}{\Delta k_s} = -\pi\frac{\delta\epsilon_s}{\Delta\epsilon_s}, \tag{A6}$$

a result which is interesting in itself because it shows that, for the immersed system, the phase-shift can be interpreted as an energy-shift $\delta\epsilon_s$, measured in units of the free-energy difference $\Delta\epsilon_s$. For our purposes, however, the interest of (A6) is that it can be inserted in (A3) to yield

$$T = \sum_s g'(\epsilon_s)\delta\epsilon_s = \frac{-1}{\pi}\sum_s g'(\epsilon_s)\eta_s\Delta\epsilon_s, \tag{A7}$$

and since the $\Delta\epsilon_s$ are just free energy differences, Fermi's golden rule can be used to pass to the continuum limit and obtain

$$T = \frac{-1}{\pi}\int g'(\epsilon)\eta(\epsilon)\,d\,\epsilon. \tag{A8}$$

Using partial integration, with $\eta(\infty) = 0$ (because for large energies the potential becomes unimportant) but $\eta(0)$ not necessarily zero, one then has

$$T = \frac{1}{\pi}\int g(\epsilon)\eta'(\epsilon)\,d\,\epsilon + \frac{1}{\pi}g((0)\eta(0). \tag{A9}$$

By using the identify $\chi'(\epsilon) = \delta(\epsilon)$ the expression (A9) may be written in the form

$$T = \frac{1}{\pi}\int g(\epsilon)\left(\frac{\partial}{\partial\epsilon}\eta(\epsilon)\chi(\epsilon)\right)\,d\,\epsilon, \tag{A10}$$

and when written in this form it may be compared with (A2) to give

$$d\,E(\epsilon) = d(\eta(\epsilon)\chi(\epsilon)), \tag{A11}$$

as required.

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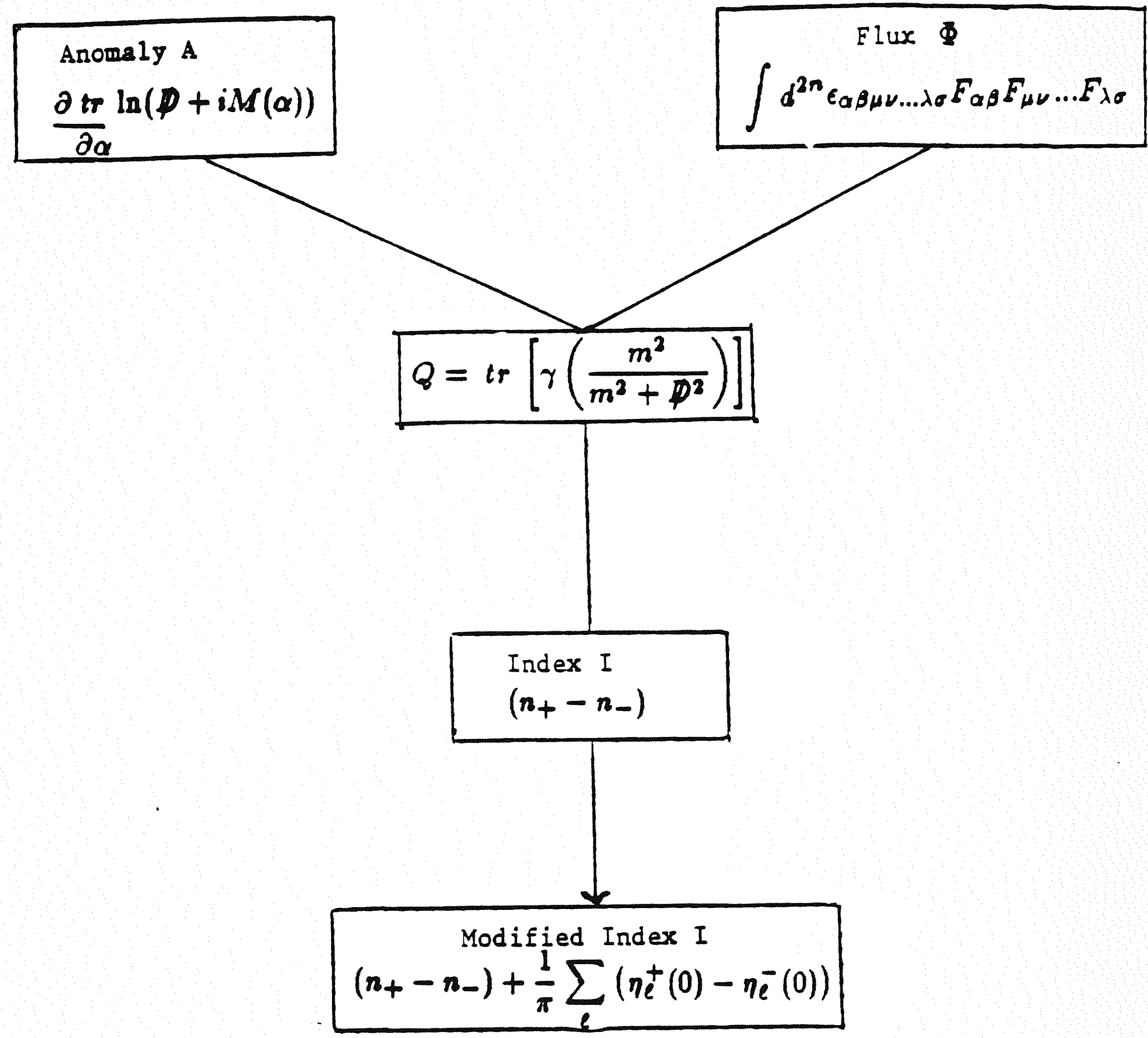
Anomaly A
$\frac{\partial\, tr \ln(\not{D} + iM(\alpha))}{\partial\alpha}$
Flux Φ
$\int d^{2n}\epsilon_{\alpha\beta\mu\nu\ldots\lambda\sigma}F_{\alpha\beta}F_{\mu\nu}\ldots F_{\lambda\sigma}$
$Q = tr\left[\gamma\left(\frac{m^2}{m^2+\not{D}^2}\right)\right]$
Index I
$(n_+ - n_-)$
Modified Index I
$(n_+ - n_-) + \frac{1}{\pi}\sum_\ell (\eta_\ell^+(0) - \eta_\ell^-(0))$

FIG. 1

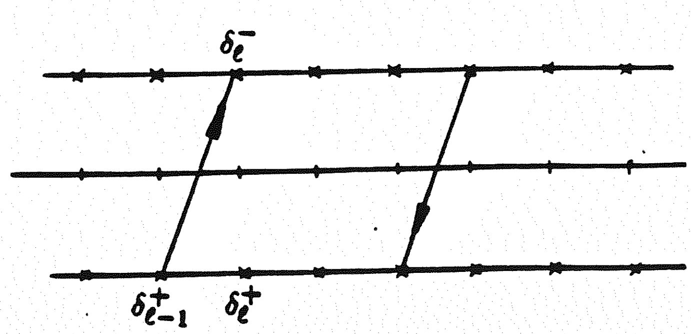
δ_ℓ^-
$\delta_{\ell-1}^+$
δ_ℓ^+

FIG. 2