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INTEGRABLE SOLUTIONS OF THE HIERARCHY OF THE BBGKY-TYPE FOR BROWNIAN PARTICLES IN THE MEAN-FIELD LIMIT

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Introduction

Let us consider the gradient hierarchy of the BBGKY-type for correlation functions $\rho_t(X_m)$, $X_m = (x_1,\dots,x_m) \in \mathbb{R}^{dm}$, $x_j \in \mathbb{R}^d$, of a nonequilibrium system of diffusing particles, interacting via a pair, integrable smooth potential $\varepsilon\,\Phi(x)$. Let us assume also that the following asymptotic relation holds

$$\rho_t(X_m) = \varepsilon^{-m}\,\rho_t^{\varepsilon}(X_m),\ \varepsilon \searrow 0$$

where the functions $\rho_t^{\varepsilon}(X_m)$ have a limit when $\varepsilon \searrow 0$. Then

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the hierarchy is written as follows

(1)

$$\frac{\partial}{\partial t} \rho_t^\varepsilon(X_m) = \sum_{j=1}^{m} \frac{\partial}{\partial x_j} \Big\{ \beta^{-1} \frac{\partial}{\partial x_j} \rho_t^\varepsilon(X_m) + \varepsilon \, \rho_t^\varepsilon(X_m) \frac{\partial}{\partial x_j} U(X_m)$$

$$+ \int_{\mathbb{R}^d} \Big(\frac{\partial \phi}{\partial x_j}\Big)(x_j - x_{m+1}) \, \rho_t^\varepsilon(X_{m+1}) \, dx_{m+1} \Big\}$$

where β is the inverse temperature and $\varepsilon U(X_m)$ is the potential energy

$$U(X_m) = \sum_{1 \le i < j \le m} \phi(x_i - x_j) \; ,$$

$$\frac{\partial}{\partial x_j}\Big(f \frac{\partial h}{\partial x_j} \Big) = \sum_{\nu=1}^{d} \Big\{ \frac{\partial f}{\partial x_j^\nu} \frac{\partial h}{\partial x_j^\nu} + f \frac{\partial^2 h}{\partial (x_j^\nu)^2} \Big\} \; .$$

In the mean-field limit ($\varepsilon \searrow 0$) the considered hierarchy is transformed into a hierarchy of the Vlasov type

(2)

$$\frac{\partial}{\partial t} \rho_t^0(X_m) = \sum_{j=1}^{m} \frac{\partial}{\partial x_j} \Big\{ \beta^{-1} \frac{\partial}{\partial x_j} \rho_t^0(X_m) +$$

$$+ \int_{\mathbb{R}^d} \Big(\frac{\partial \phi}{\partial x_j}\Big)(x_j - x_{m+1}) \, \rho_t^0(X_{m+1}) \, dx_{m+1} \Big\}$$

In this paper we propose a justification of the mean-field limit in a class of integrable correlation functions satisfying eq.(1) in a weak sense. We introduce a semigroup π_{ε}^{t} in a Banach space $\mathbb{L}_{\xi}^{1}$ of sequences of symmetric itegrable functions and prove that the sequense

$$\rho_{\varepsilon}^{t}(X_m) = (\pi_{\varepsilon}^{t}\rho^{\varepsilon})(X_m), \quad \rho^{\varepsilon} \in \mathbb{L}_{\xi}^{1}$$

satisfies eq.(1) in a weak sense.We show that, if the sequence $\exp(\frac{1}{2}\varepsilon U(X_m))\,\rho_t^{\varepsilon}(X_m)$, $m\geq 1$, belongs to $\mathbb{L}_{\xi}^{1}$, then $\rho_t^{\varepsilon}(X_m)$ converges weakly to

$$\rho_t^{o}(X_m) = (\pi_{o}^{t}\rho^{o})(X_m), \quad m\geq 1,$$

where π_{o}^{t} is defined as a map

$$\mathbb{L}_{\xi}^{1} \to \mathbb{L}_{\xi(t)}^{1}, \quad \text{if} \quad \xi(t) = \sqrt{2}\,\exp\{\hat{\nu}(t)\}\,\xi < 1 .$$

We also prove that the sequence ρ_t^{o} is a weak solution of eq.(2).

The norm in the Banach space $\mathbb{L}_{\xi}^{1}$ is defined as follows

$$\|\Psi\|_{\mathbb{L}_{\xi}^{1}} = \max_{n\geq 1} \xi^{-n} \|\Psi_n\|_{L^1(\mathbb{R}^{dn})} .$$

The mean-field limit in a mechanical and a special random mechanical systems was studied earlier, respectively in [2,3](see also [4,5]). The mean-field limit for eq.(1) in a class of bounded correlation functions is investigated in [6]. It is found in [7] that there is a space in which a solution of eq.(2) exists and is unique.

I. Main theorem.

Let $P_\varepsilon^t(X_n;Y_n)$ be a fundamental solution of the n-particle Smoluchowski equation (eq.(1) without the integral term in the r.s.). The operators $P_{\varepsilon,n}^t$, $t\geq o$,

$$(P_{\varepsilon,n}^t\psi)(X_n) = \int_{\mathbb{R}^{dn}} P_\varepsilon^t(X_n;Y_n)\psi(Y_n)dY_n$$

defines a contraction strongly continuous semigroup in $L^1(\mathbb{R}^{dn})$. Let P_ε^t be a diogonal operator in $\mathbb{L}_\xi^1$, given by

$$(P_\varepsilon^t\psi)(X_n) = (P_{\varepsilon,n}^t\psi)(X_n).$$

It is evident that P_ε^t is a contraction strongly continuous semigroup in $\mathbb{L}_\xi^1$. Let us define an operator $\int d_x$ which is bounded in $\mathbb{L}_\xi^1$

$$(\int d_x\psi)(X_n) = \int_{\mathbb{R}^d} \psi(x,X_n)dx$$

Then π_ε^t

(1.1) $$\pi_\varepsilon^t = \exp\{\varepsilon^{-1}\int d_x\} P_\varepsilon^t \exp\{-\varepsilon^{-1}\int d_x\}$$

is a strongly continuous semigroup in $\mathbb{L}_\xi^1$. Its structure coincide with a structure of an evolution operator of the BBGKY-hierarchy [8].

Lemma 1.1

The sequence $\rho_t^\varepsilon = \pi_\varepsilon^t\rho^\varepsilon$, $\rho^\varepsilon\in \mathbb{L}_\xi^1$,is a weak solution of eq.(1), that is

$$(1.2) \qquad \frac{\partial}{\partial t} \int_{dm} \rho_t^{\varepsilon}(X_m)\, h(X_m)\, dX_m =$$

$$= \sum_{j=1}^{m} \int_{\mathbb{R}^{dm}} \Big\{ \beta^{-1} \rho_t^{\varepsilon}(X_m) \frac{\partial^2}{\partial x_j^2} h(X_m) -$$

$$- \Big(\frac{\partial}{\partial x_j} h(X_m)\Big) \Big[\varepsilon\, \rho_t^{\varepsilon}(X_m) \frac{\partial}{\partial x_j} U(X_m) +$$

$$+ \int_{\mathbb{R}^d} \Big(\frac{\partial \phi}{\partial x_j}\Big)(x_j - x_{m+1})\, \rho_t^{\varepsilon}(X_{m+1})\, dx_{m+1} \Big] \Big\}\, d X_m .$$

Lemma 1.2

Let $P_x(d\tilde{x})$ be the Wiener measure on $\Omega_d = (\dot{\mathbb{R}}^d)^{\mathbb{R}^+}$ and $P_{X_m}(d\tilde{X}_m) = \prod_{j=1}^{m} P_{x_j}(d\tilde{x})$. Let $*$ be an operation of multiplication, defined on sequences

$$\{ \psi(X_n, \tilde{X}_n),\ X_n \in \mathbb{R}^{dn},\ \tilde{X}_n \in \Omega_d^n \}_{n \geq o}, \quad \psi(\emptyset,\emptyset) = 1$$

by

$$(\psi_1 * \psi_2)(X_n, \tilde{X}_n) = \sum_{\{s\} \in (1,..n)} \psi_1(X_{\{s\}}, \tilde{X}_{\{s\}}) \psi_2(X_{\{n\backslash s\}}, \tilde{X}_{\{n\backslash s\}})$$

$$\{n\backslash s\} = (1,\dots,n)\backslash\{s\} .$$

Let $(\psi)^{-1}$ be the inverse element to ψ with respect to $*$ and

$$(D_{(X_m;\tilde{X}_m)} \psi)(X_n'; \tilde{X}_n') = \psi(X_m, X_n'; \tilde{X}_m, \tilde{X}_n')$$

There exists a measurable function $\hat{U}_t^{\varepsilon}(X_n,\tilde{X}_n)$ such that, if

$$\Pi_{\varepsilon}(X_m,\tilde{X}_m|X_n',\tilde{X}_n') =$$

$$= \varepsilon^{-n}\left((\exp\{ -\beta\ \hat{U}_t^{\varepsilon} \})^{-1} * D_{(X_m;\tilde{X}_m)} \exp\{ -\beta\ \hat{U}_t^{\varepsilon} \} \right)(X_n';\tilde{X}_n') \quad ,$$

where the n-th component of the sequence $\exp\{ -\beta\ \hat{U}_t^{\varepsilon} \}$ equals $\exp\{ -\beta\ \hat{U}_t^{\varepsilon}(X_n;\tilde{X}_n)$,then the cluster expansion for π_{ε}^t holds

(1.3)

$$(\pi_{\varepsilon}^t\ \rho_t^{\varepsilon})(X_m) = \sum_{n\ge o} \frac{1}{n!} \int_{\mathbb{R}^{dn}} dX_n' \int_{\Omega_d^{n+m}} P_{X_m,X_n'}(d\tilde{X}_m d\tilde{X}_n') \times$$

$$\times\ \Pi_{\varepsilon}(X_m,\tilde{X}_m|X_n,\tilde{X}_n')\ \rho(\tilde{X}_m(t\beta^{-1}),\tilde{X}_n'(t\beta^{-1})) \ .$$

Lemma 1.3

If the potential ϕ is a positive-definite function from $C^3(\mathbb{R}^d)$ and

$$|\phi(x)| \le \phi^o,\quad |\nabla\phi(x)| \le \phi^o,\quad |\Delta\phi(x)| \le \phi^o,\quad \Delta = \nabla^2, \nabla=\frac{\partial}{\partial x} \ .$$

then the following uniform in ε bound holds

(1.4)

$$\operatorname*{ess\,sup}_{all(x,x',\tilde{x},\tilde{x}')} \exp\{-\tfrac{1}{2}\ \beta\ \hat{U}_t^{\varepsilon}(X_m,X_n';\tilde{X}_m,\tilde{X}_n') \}\ |\Pi_{\varepsilon}(X_m,\tilde{X}_m|X_n,\tilde{X}_n')| \le$$

$$\le\ n!\left(\sqrt{2}\ \exp\{ \hat{\nu}(t) \} \right)^{m+n} ,$$

$$\hat{\nu}(t) = (\beta\ \phi^o\ \varkappa(t))^2 + \tfrac{1}{2}\ \phi(0) + \tfrac{1}{2}\ t\ (-\Delta\phi)(0) \ ,$$

and the functions $\Pi_{\varepsilon}(X_m,\tilde{X}_m|X_n',\tilde{X}_n')$ converge a.e. to functions

$\Pi_o(X_m,\tilde{X}_m | X_n',\tilde{X}_n')$, satisfying (1.4) for $\varepsilon = o$.

<u>Corollary</u> .The operator π_o^t, defined by (1.3) for $\varepsilon = o$ maps $\mathbb{L}_\xi^1$ into $\mathbb{L}_{\xi(t)}^1$, if $\xi(t) < 1$.

Theorem 1.1

Let the conditions of the Lemma 1.3 be satisfied.If the following conditions are also satisfied

$$\exp \{ \tfrac{1}{2} \beta U \} \rho^\varepsilon \in \mathbb{L}_\xi^1 ,$$

$$\| \exp \{ \tfrac{1}{2} \beta U \} \rho^\varepsilon - \rho^o \|_{\mathbb{L}_\xi^1} \leq o(\varepsilon) ,$$

where $(\exp \{ \tfrac{1}{2} \beta U \})(X_n) = \exp \{ \tfrac{1}{2} \beta U(X_n) \}$,

then the functions $(\pi_\varepsilon^t \rho^\varepsilon)(X_m)$ converge weakly to the functions $(\pi_o^t \rho^o)(X_m)$, sequence of which satisfies eq.(2) in a weak sense

(1.5) $$\frac{\partial}{\partial t} \int_{\mathbb{R}^{dm}} \rho_t^o(X_m)\, h(x_m)\, dX_m =$$

$$= \sum_{j=1}^{m} \int_{\mathbb{R}^{dm}} \Big\{ \beta^{-1} \rho_t^o(X_m) \frac{\partial^2}{\partial (x_j)^2} h(X_m) -$$

$$- \Big(\frac{\partial}{\partial x_j} h(X_m) \Big) \int_{\mathbb{R}^d} \Big(\frac{\partial \phi}{\partial x_j} \Big)(x_j - x_{m+1})\ \rho_t(X_{m+1})\, dx_{m+1} \Big\}\, dX_m .$$

2. Cluster expansion .

Let us prove the equality (1.3). We start from resumming in (1.1)

(2.1)

$$\pi_\varepsilon^t = \sum_{n\geq o} \frac{\varepsilon^{-n}}{n!} \int_{\mathbb{R}^{dn}} dX_n' \sum_{s=o}^{n} \frac{n!}{s!(n-s)!} (-1)^{n-s} D_{X_s'} P^t D_{X'_{\{n\setminus s\}}}$$

where $(D_{X_m} \psi)(X_n') = \psi(X_m, X_n')$.

Let $*$ be the operation of multiplication defined on sequences of kernels $\{ K(X_n;Y_n) \}_{n\geq o}$, $K(\emptyset,\emptyset) = 1$ (see Lemma 1.2) . $e_o = (1,0,0,\dots)$ is the unit with the respect to $*$. Define

$$(K)^{-1} = \sum_{n\geq o} (-1)^n \underbrace{(K-e_o)*\dots*(K-e_o)}_{n} , \quad (K)^{-1}* K = e_o ,$$

$$<K>_\psi = \sum_{n\geq o} \frac{1}{n!} \int_{\mathbb{R}^{dn}} dX_n \int_{\mathbb{R}^{dn}} K(X_n;Y_n) \Psi(Y_n)\, dY_n$$

Then (1.2) is rewritten in the following fashion

(2.2)

$$(\pi_\varepsilon^t \rho^\varepsilon)(X_m) = \int_{\mathbb{R}^{dm}} < e_{-1}* D_{(X_m;Y_m)} P_\varepsilon^t >_{\hat{\varepsilon}^{-1} D_{Y_m} \rho^\varepsilon} dY_m$$

where $(e_{\pm 1})(X_n;Y_n) = (\pm 1)^n \delta(X_n-Y_n) , (\hat{\varepsilon}^{-1})(\rho^\varepsilon)(X_n)=\varepsilon^{-n}\rho^\varepsilon(X_n)$

As a result we derive the cluster expansion

(2.3)

$$(\pi_\varepsilon^t \rho^\varepsilon)(X_m) = \int_{\mathbb{R}^{dm}} < (P_\varepsilon^t)^{-1}* D_{(X_m;Y_m)} P_\varepsilon^t >_{\hat{\varepsilon}^{-1} D_{Y_m} \rho^\varepsilon} dY_m$$

(2.3) follows from the law of conservation of probability for the Smoluchowski equation and the equality $e_{-1}=(e_{+1})^{-1}$

in the following way

$$<(P_\varepsilon)^{-1} * D_{(X_m;Y_m)} P_\varepsilon >_\psi = \sum_{n\geq o} <(P_\varepsilon^t - e_o) * \underset{n}{\ldots} * (P_\varepsilon^t - e_o) * D_{(X_m;Y_m)} P_\varepsilon^t >_\psi =$$

$$= \sum_{n\geq o} <(e_1 - e_o) * \underset{n}{\ldots} * (e_1 - e_o) * D_{(X_m;Y_m)} P_\varepsilon^t >_\psi =$$

$$= < (e_{+1})^{-1} * D_{(X_m;Y_m)} P_\varepsilon^t >_\psi = < e_{-1} * D_{(X_m;Y_m)} P_\varepsilon^t >_\psi .$$

It is well known that, if a function $\mu_t^\varepsilon(X_n)$ satisfies the Smoluchowski equation, the function

$$\exp \{ - \tfrac{1}{2} \varepsilon\beta \, U(X_n) \} \, \mu_t^\varepsilon(X_n)$$

satisfies the heat equation with the potential $V_\varepsilon(X_n)$

$$V_\varepsilon(X_n) = \tfrac{1}{2} \beta\varepsilon \sum_{j=1}^{n} \left\{ \frac{\partial^2}{\partial x_j^2} U(X_n) - \tfrac{1}{2} \beta\varepsilon \left(\frac{\partial}{\partial x_j} U(X_n) \right)^2 \right\} .$$

Applying the Feynman-Kac formula we derive the following representation for the kernel P_ε^t

(2.4) $$P_\varepsilon^t(X_n;Y_n) = \exp \{ - \tfrac{1}{2} \beta\varepsilon \, (U(X_n) - U(Y_n)) \} \times$$

$$\times \int_{\Omega_d^n} P_{X_n}(d\tilde{X}_n) \exp \left\{ \int_o^{t\beta^{-1}} V_\varepsilon(\tilde{X}_n(\tau)) d\tau \right\} \delta(\tilde{X}_n(t\beta^{-1}) - Y_n) .$$

(2.2) and (2.4) yield (1.3) with

$$\hat{U}_t^\varepsilon(X_n;\tilde{X}_n) = \tfrac{1}{2} \varepsilon \{ U(X_n) - U(\tilde{X}_n(t\beta^{-1})) \} + \beta^{-1} \int_o^{t\beta^{-1}} V(\tilde{X}_n(\tau)) d\tau .$$

3. Proof of the main estimate .

In order to prove (1.4) we shall make use of the following identity

$$(3.1) \qquad \exp\left\{-\frac{1}{4}\beta^2\varepsilon^2\sum_{j=1}^{n}\int_{o}^{t\beta^{-1}}\left(\frac{\partial U}{\partial x_j}\right)^2(\tilde{X}_n(\tau))\,d\tau\right\} =$$

$$= \int_{\Omega_d^n} P(X_n^*)\exp\left\{-\tfrac{1}{2}\beta\varepsilon i\sum_{j=1}^{n}\int_{o}^{t\beta^{-1}}\left(\left(\frac{\partial U}{\partial x_j}\right)(\tilde{X}_n(\tau)),dx_j^*(\tau)\right)\right\} .$$

where $P(X_n^*) = \prod_{j=1}^{n} P(dx_j^*)$, $P(dx^*) = P_o(dx^*)$, $P_o(dx^*)$ is the Wiener measure , $\int(\ldots,dx^*(\tau))$ is the stochastic integral, $(\ldots,\ \ldots)$ is the scalar product of $\mathbb{E}^d$ (we omit ~ over x_j in derivatives).

Substituting (3.1) into (1.3) we obtain

(3.2)

$$\Pi_\varepsilon(X_m,\tilde{X}_m|X_n',\tilde{X}_n') = \int_{\Omega_d^{m+n}} P(dX_m^*)P(dX_n'^*)\ \Pi_\varepsilon(\hat{X}_m|\hat{X}_n') ,$$

where $\hat{X}_m = (\ \hat{x}_1,\ldots,\hat{x}_m\ )$, $\hat{x}_j = (\ x_j,\tilde{x}_j,x_j^*\ ) \in \mathbb{R}^d \times \Omega_d^2$,

$$\Pi_\varepsilon(\hat{X}_m|\hat{X}_n') = \varepsilon^{-n}((\ \exp\{-\varepsilon\beta\ \hat{U}_t\ \})\ *\ D_{\hat{X}_m}\exp\{\ -\varepsilon\beta\ \hat{U}_t\ \}\ )(\hat{X}_n')$$

$$\hat{U}_t(\hat{X}_n) = \sum_{1\le k<j\le n} \hat{\phi}_t(\hat{x}_k|\hat{x}_j) ,$$

$$\hat{\phi}_t(\hat{x}_k|\hat{x}_j) = \tilde{\phi}_t(x_k-x_j,\tilde{x}_k-\tilde{x}_j) + \phi_t^*(\tilde{x}_k-\tilde{x}_j|x_k^*,x_j^*) ,$$

$$\tilde{\phi}(x,\tilde{x}) = \tfrac{1}{2}\left[\ \phi(x) - \phi(\tilde{x}(t\beta^{-1})) + \int_{o}^{t\beta^{-1}}(-\Delta\phi)(\tilde{x}(\tau))\,d\tau\ \right] ,$$

$$\phi^*_t(\tilde{x}|x^*_k,x^*_j) = \tfrac{1}{2}\, i\{ \varphi_t(\tilde{x}|x^*_k) - \varphi_t(\tilde{x}|x^*_j) \} ,$$

$$\varphi_t(\tilde{x}|x^*) = \tfrac{1}{2} \int_0^{t\beta^{-1}} ((\nabla\phi)(\tilde{x}(\tau)),\, dx^*(\tau)) ,$$

The sequence $\Pi_\varepsilon(\hat{X}_m|\hat{X}'_n)$ $m\geq 1$, $n\geq 1$ satisfies the standard relation [9]

(3.3) $$\Pi_\varepsilon(\hat{X}_m|\hat{X}'_n) = \exp\{ -\varepsilon\beta \sum_{\substack{l=1,\\ l\neq j}}^{m} \hat{\phi}_t(\hat{x}_l|\hat{x}_j) \} \times$$

$$\times \sum_{\{s\}\in(1,..n)} K_\varepsilon(\hat{x}_j|\hat{X}'_{\{s\}})\, \Pi_\varepsilon(\hat{X}_{m(j)},\hat{X}'_{\{s\}}|\hat{X}'_{\{n\setminus s\}}) ,$$

$\Pi_\varepsilon(\hat{X}_m|\emptyset) = \exp\{ -\varepsilon\beta\, \hat{U}_t(\hat{X}_m) \}$, $\Pi_\varepsilon(\emptyset|\hat{X}_n) = 0$, $m(j)=(1,..m)\setminus j$

$$K_\varepsilon(\hat{x}|\hat{X}_n) = \prod_{j=1}^{n} \varepsilon^{-1}(\exp\{ -\varepsilon\beta\hat{\phi}_t(\hat{x}|\hat{x}_j) \}- 1) .$$

(3.3) has the limit for $\varepsilon \to 0$.

(3.4) $$\Pi_o(\hat{X}_m|\hat{X}'_n) =$$

$$= \sum_{\{s\}\in(1,..n)} \prod_{l\in\{s\}} (-\beta)\, \hat{\phi}_t(\hat{x}_j|\hat{x}_l)\, \Pi_o(\hat{X}_{m(j)},\hat{X}'_{\{s\}}|\hat{X}'_{\{n\setminus s\}}) .$$

Proposition 1.3 (The main bound)

If the conditions of Lemma 1.3 are satisfied then the following uniform in all variables,except t, bound

(3.5)

$$\left\{ \int_{\Omega_d^m} P(dX^*_m) \left[\int_{\Omega_d^n} P(dX^*_n{}') \; |\Pi_\varepsilon(\hat{X}_m|\hat{X}'_n)| \right]^2 \right\}^{\frac{1}{2}} \leq n!(\sqrt{2}\,\exp\{\hat{\nu}(t)\})^{m+n}$$

Proof.

To derive (3.5) we have to symmetrize (3.3), taking into consideration that the potential

$$\phi_{ot}(x,x) = \tilde{\phi}_t(x,\tilde{x}) + \tfrac{1}{2}\phi(\tilde{x}(t\beta^{-1}))$$

is stable (see also [6,9,10]). Now we prove (3.5) by induction for $\varepsilon = 0$.

Let us integrate (3.4) by $P(dX_n^{'*})$ and apply the Schwartz inequality

$$\int_{\Omega_d^{<s>}} P(dX_{\{s\}}^{'*}) \left| \prod_{l\in\{s\}} \hat{\phi}_t(\hat{x}_j|\hat{x}_l) \right| \times$$

$$\times \int_{\Omega_d^{<n-s>}} P(dX_{\{n\backslash s\}}^{'*}) \left| \Pi_o(\hat{X}_{m(j)},\hat{X}_{\{s\}}^{'}| \hat{X}_{\{n\backslash s\}}^{'}) \right|$$

$$\leq \left[\int_{\Omega_d^{<s>}} P(dX_{\{s\}}^{'*}) \left| \prod_{l\in\{s\}} \hat{\phi}_t(\hat{x}_j|\hat{x}_l) \right|^2 \right]^{\frac{1}{2}} \times$$

$$\times\left[\int_{\Omega_d^{<s>}} P(dX_{\{s\}}^{'*}) \left[\int_{\Omega_d^{<n\backslash s>}} P(dX_{\{n\backslash s\}}^{'*}) |\Pi_o(\hat{X}_{m(j)},\hat{X}_{\{s\}}^{'}|\hat{X}_{\{n\backslash s\}}^{'})| \right]^2 \right]^{\frac{1}{2}}$$

where <s> is the number of elements in the sequence {s}. Let us square the obtained inequality, split the sum over {s} into two sums (over $\{s_1\}$ and $\{s_2\}$) and integrate by $P(dX_m^*)$ the resulting expression, utilizing the Schwartz inequality. Assume that (3.5) holds for all $<s> \leq m+n-1$.In this case

$$\int_{\Omega_d^m} P(dX_m^*) \left[\int_{\Omega_d^n} |\Pi_o(\hat{X}_m|\hat{X}_n)| \; P(dX_n^{'*}) \right] \leq (n! \sum_{s\geq o} \frac{1}{s!} K_{s,o})^2 \times$$

$$\times \; (\sqrt{2} \exp \{ \hat{\nu}_o(t) \})^{2(m+n-1)}$$

$$K_{s,o} = \beta^{s} \operatorname*{ess\,sup}_{\text{all}(x,\tilde{x})} \int_{\Omega_d} P(dx^*) \left(\int_{\Omega_d^s} P(dX_s^*) \left| \prod_{l=1}^{s} \phi_t(\hat{x}|\hat{x}_l) \right|^2 \right)^{\frac{1}{2}}.$$

Making use of the Schwartz and generalized Helder inequalities we obtain the following bound for $K_{s,o}$

$$K_{s,o} \le \beta^{s} \operatorname*{ess\,sup}_{\text{all}(x,\tilde{x})} \left[\int_{\Omega_d^s} P(dX_s^*) \prod_{l=1}^{s} \int_{\Omega_d} P(dx^*) |\hat{\phi}_t(\hat{x}|\hat{x}_l)|^2 \right]^{\frac{1}{2}} \le$$

$$\le \beta^{s} \operatorname*{ess\,sup}_{\text{all}(x,\tilde{x})} \left(\prod_{l=1}^{s} \int_{\Omega_d^2} P(dx_l^*)\, P(dx^*) |\hat{\phi}_t(\hat{x}|\hat{x}_l)|^{2s} \right)^{\frac{1}{2s}}.$$

It is well known that

$$\int_{\Omega} P(dx^*) \left[\int_0^{t\beta^{-1}} ((\nabla\phi)(\tilde{x}(\tau), dx^*(\tau)) \right]^{2s} = \frac{(2s)!}{(s)!} \left[\int_0^{t\beta^{-1}} (\nabla\phi)^2 (\tilde{x}(\tau)) d\tau \right]^{\frac{1}{2}}$$

With the help of the inequalities

$$(a^s + b^s)^{\frac{1}{s}} \le (a+b) \ , \ (a+b)^s \le 2^s (a^s + b^s) \ , \ 3^{-n} n^n \le n! \le n^n$$

we have

$$K_{s,o} \le (s!)^{\frac{1}{2}} \left\{ 72\beta \left(|\tilde{\phi}_t|_o + 2|\nabla\phi|_o t\beta^{-1} \right) \right\}^s \le (s!)^{\frac{1}{2}} (\phi^o \beta \varkappa_o(t))^s$$

where

$$|\phi|_o = \operatorname*{ess\,sup}_{x\in X} |\phi(x)| \ , \varkappa_o(t) = 72(1+\tfrac{1}{2}\ 5t\beta^{-1}), \hat{\nu}_o = (\varkappa_o(t)\beta\phi^o)^2.$$

Now it is not difficult to prove the proposition for $\varepsilon > o$, using the symmetrized (3.3)[6] and the above arguements. As a result

$$\varkappa(t) = \exp\{ \beta\phi^o (1+\tfrac{1}{2}\ t\beta^{-1}) \} \varkappa_o(t) \ .$$

Now we return to the Theorem 1.1. At first we consider the simplest case

$$\rho^{\varepsilon} = \exp \{ -\tfrac{1}{2} \varepsilon\beta U \} \rho^{0} .$$

Since the algebraic structure of the functions $\Pi_{\varepsilon}(\hat{X}_m|\hat{X}'_n)$ are known , they converge to the functions $\Pi_{0}(\hat{X}_m|\hat{X}'_n)$ a.e. . From the Lebesque theorem it follows that

$$\int_{\mathbb{R}^{dn}} (\pi^{t}_{\varepsilon} \rho^{\varepsilon})(X_m) \ h(X_m) \ dX_m \underset{\varepsilon\to 0}{\to} \int_{\mathbb{R}^{dn}} (\pi^{t}_{0} \rho^{0})(X_m) \ h(X_m) \ dX_m .$$

It means that the r.s. of (1.2) converges to the r.s. of (1.5) . Since the r.s. of (1.5) is a continuous function of t (the series in (1.3) converges uniformely on a finite time interval) the l.s. of (1.2) converges to the left side of (1.5) . It is clear that in a general case

$$\rho^{\varepsilon} = \exp \{ -\tfrac{1}{2} \varepsilon\beta \ U \} (\rho^{0} + \rho'^{\varepsilon}) .$$

From Lemmas 1.1 , 1.2 it follows that the remaining term $\pi^{t}_{\varepsilon}\exp\{ -\tfrac{1}{2}\varepsilon\beta \ U \} \rho'^{\varepsilon}$ converges to zero in $\mathbb{L}^{1}_{\xi}$ if

$$\xi(t) = \sqrt{2} \exp \{ \hat{\nu}(t) \} \xi < 1 .$$

The theorem is proved .

4. Proof of Lemma 1.1

Let us put

$$\rho^{\varepsilon}_{N} = \exp\{ \varepsilon \int d_x \} \mu^{\varepsilon}_{N} \quad , \mu^{\varepsilon}_{N} \in \mathbb{L}^{1}_{\xi} \ , \ \mu^{\varepsilon}_{N}(X_n) = 0 \ , \ n > N .$$

Then

$$\rho^{\varepsilon}_{t,N}(X_m) = (\pi^{t}_{\varepsilon}\ \rho^{\varepsilon}_{N})(X_m) = (\exp\{\ \int d_x\ \} P^{t}_{\varepsilon}\ \mu^{\varepsilon}_{N})(X_m) =$$

$$= \sum_{n=o}^{N-m} \frac{1}{n!} \int_{\mathbb{R}^{dm}} dX_n'\ (P^{t}_{\varepsilon}\ \mu^{\varepsilon}_{N})(X_m, X_n') \ .$$

It can be shown that $\rho^{\varepsilon}_{t,N}$ satisfies (1) in a classical sense. Now let us differentiate it ,taking into consideration that ($P^{t}_{\varepsilon,n}\ \mu^{\varepsilon}{}_{N})(X_n) = \mu^{\varepsilon}_{t}(X_n)$ satisfies the Smoluchowski equation

$$\frac{\partial}{\partial t} \rho(X_m) = \sum_{j=1}^{m} \frac{\partial}{\partial x_j} \left\{ \beta^{-1}\frac{\partial}{\partial x_j} \rho^{\varepsilon}_{t,N}(X_m) + \rho^{\varepsilon}_{t,N}(X_m)\frac{\partial}{\partial x_j} U(X_m) + \right.$$

$$+ \varepsilon \sum_{n=o}^{N-m} \frac{n}{n!}\ \varepsilon^{-n} \int_{\mathbb{R}^{dn}} (\frac{\partial\ \phi}{\partial\ x_j})(x_j - x_{m+1})\mu^{\varepsilon}_{t}(X_m, X_n')\,dX_n$$

The derivatives in the inner variables(X_n') disappeared since

$$\int_{\mathbb{R}} (\frac{\partial\ h}{\partial\ x})(x)dx = 0\ ,\ f\ ,\ \frac{\partial\ h}{\partial\ x} \in L^1(\mathbb{R})\ .$$

and $\mu^{\varepsilon}_{t}(X_n) \in L^1 \cap C^2$[10]. As a result the last term is equal

$$\int_{\mathbb{R}^d} (\frac{\partial\ \phi}{\partial\ x_j})(x_j - x_{m+1}) \left[\sum_{n=o}^{N-m-1} \frac{\varepsilon^{-n}}{n!} \int_{\mathbb{R}^{dn}} \mu^{\varepsilon}_{t}(X_m, X_n')\ dX_n' \right] dx_{m+1} =$$

$$= \int_{\mathbb{R}^d} (\frac{\partial\ \phi}{\partial\ x_j})(x_j - x_{m+1})\ \rho^{\varepsilon}_{t,N}(X_{m+1})\ dx_{m+1}\ ,\ m \le N.$$

Hence the sequence $\{\ \rho^{\varepsilon}_{t,N}(X_m)\ \}_{m<N}$ satisfies (1) in a weak sense. Let $N \to \infty$ .Then $\rho^{\varepsilon}_{t,N}$ converges to ρ^{ε}_{t} in the topology of $\mathbb{L}^1_{\xi}$. By the Lebesque theorem the r.s. of (1.2) for $\rho^{\varepsilon}_{t,N}$ converges to the r.s. of (1.2). Since π^{t}_{ε} is a strongly continuous semigroup the same is true for the

corresponding left sides .The proof is complete .

REMARK . Our theorem does not allow the canonical correlation functions to converge in the mean-field limit since their limit satisfies the compatibility condition

$\int_{\mathbb{R}^d} \rho_t(X_n)dx_n = \rho_t(X_{n-1})$ .and $\rho_t \in \mathbb{L}^1_\xi$ only if $\xi \geq 1$.

But there is a possibility to improve our bounds in such a way that $\xi(t)$ goes to 0 when either t or ϕ^o goes to 0 .

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