

Title	The Wu-Yang factor and the non-Abelian Aharonov-Bohm experiment
Creators	Horváthy, P. A.
Date	1986
Citation	Horváthy, P. A. (1986) The Wu-Yang factor and the non-Abelian Aharonov-Bohm experiment. (Preprint)
URL	<a href="https://dair.dias.ie/id/eprint/836/">https://dair.dias.ie/id/eprint/836/</a>
DOI	DIAS-STP-86-04

The Wu-Yang factor and the non-Abelian Aharonov-Bohm experiment

P.A. Horváth

School of Theoretical Physics  
Dublin Institute for Advanced Studies  
10 Burlington Road, Dublin 4  
Ireland

Abstract

The scattering of nucleons on a non-Abelian flux is shown to depend, as predicted by Wu and Yang, on the non-integrable phase factor.

PACS numbers: 11.15.-q, 11.30.Jw

The cornerstone of the Wu-Yang approach to gauge theory [1] is the non-integrable phase factor

$$(1) \quad \Phi = P(\exp - \oint A_i dx^i).$$

In particular, the scattering of a nucleon beam on a non-Abelian flux line-the non-Abelian version of the experiment proposed by Aharonov and Bohm for testing the role of electromagnetic potentials [2] - should depend on (1). This prediction has been confirmed recently using non-commuting path integrals [3]. The point is that such complications can be avoided. To see this, let us describe the scattering by a two-component Schrödinger equation

$$(2) \quad i \frac{\partial}{\partial t} \begin{pmatrix} \Psi \\ \varphi \end{pmatrix} = - \frac{1}{2m} (\partial_i - A_i)^2 \begin{pmatrix} \Psi \\ \varphi \end{pmatrix}$$

where  $A = A_i dx^i$  is an  $SU(2)$  gauge potential whose field strength,  $F_{jk} = \partial_j A_k - \partial_k A_j + [A_j, A_k]$ , vanishes everywhere, except at the origin. Nucleons are identified with  $SU(2)$  doublets. The validity of (2) is restricted, just like in the electromagnetic case, to the plane with the origin excluded. The remaining region is non-simply connected allowing for non-trivial values of (1).

The simplest way of confirming the Wu-Yang predictions is by paraphrasing the argument of Byers and Yang [4]. Define in fact

$$(3) \quad g(x) = P(\exp \int_{x_0}^x A_i dx^i)$$

where the integration is along an arbitrary path from a reference point  $x_0$  to  $x$  which does not cross the positive  $x$  axis. The

non-Abelian version of Stokes' theorem [5] implies [6] then that the integration is path-independent so that (3) provides us with a well-defined function.  $g(x)$  satisfies furthermore the relation  $\partial_i g = -A_i g$ . The (along the positive x-axis singular) gauge transformation  $A_i \rightarrow \bar{g}^{-1} A_i g + \bar{g}^{-1} \partial_i g$ ,  $\Psi = \bar{g}^{-1} \begin{pmatrix} \psi \\ \varphi \end{pmatrix}$ , brings (2) to a free form:

$$(4) \quad i \frac{\partial \Psi}{\partial t} = -\frac{1}{2m} \Delta \Psi$$

The new wave function becomes however double-valued: in polar coordinates  $(r, \vartheta)$ , we have

$$(5) \quad \Psi(r, 0) = \Phi \Psi(r, 2\pi).$$

Consequently, (4) admits identical solutions as long as the "boundary conditions" (5) are the same [7], i.e. for identical Wu-Yang factors. Q.E.D.

To get an explicit solution it is convenient to use another [6] (regular) gauge: since any  $SU(2)$  group element can be diagonalized,  $\Phi = \text{diag}(\exp(2\pi i \alpha), \exp(-2\pi i \alpha))$  in a suitable gauge, where the real parameter  $\alpha$  can be chosen to satisfy  $0 \leq \alpha < 1$  with no loss of generality. A gauge potential is hence

$$(6) \quad A = A_\vartheta d\vartheta = \text{diag}(\alpha, -\alpha),$$

and (2) splits hence into two, uncoupled, electromagnetic Bohm-Aharonov equations with electromagnetic potentials  $\pm(\alpha/e)d\vartheta$ :

$$(7) \quad \frac{\partial}{\partial t} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = -\frac{1}{2m} \left( \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \begin{bmatrix} \partial_\vartheta + i\alpha & 0 \\ 0 & \partial_\vartheta - i\alpha \end{bmatrix}^2 \right) \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

with periodic boundary conditions this time. (7) is solved hence

by simple applications of the Abelian results

$$(8) \begin{pmatrix} \Psi_1(x,t) \\ \Psi_2(x,t) \end{pmatrix} = \int d^2y \begin{pmatrix} K_\alpha(x,t|y,0) & 0 \\ 0 & K_{-\alpha}(x,t|y,0) \end{pmatrix} \begin{pmatrix} \Psi_1(y,0) \\ \Psi_2(y,0) \end{pmatrix}$$

where the integration is on the punctured plane. The electromagnetic BA propagator  $K_\alpha$  can be expressed as a path integral. By the well-known result ([7] and references therein):

$$(9) \quad K_\alpha = C_\alpha \sum_{n=-\infty}^{\infty} \chi^n K_n^0, \quad \chi = \exp[2\pi i \alpha],$$

where the integer  $n$  labels the homotopy classes of those paths which wind  $n$  times around the origin,  $K_n^0$  is the corresponding partial propagator defined by the free dynamics. Explicitly [8]

$$(10) \quad K_n^0(x,t|y,0) = \left(\frac{m}{it}\right) \exp\left[\frac{im}{2t}(r^2 + r'^2)\right] \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left[ik(\theta' - \theta + 2n\pi) - |k| \frac{\pi}{2}\right] J_{|k|}(mrr'/t)$$

where  $x = (r, \theta)$ ,  $y = (r', \theta')$  and  $J_m$  is the Bessel function.  $C_\alpha$  is an overall (unobservable) phase factor.

But  $\text{diag}(\chi, \chi^{-1})$  is just the Wu-Yang factor  $\Phi$ , and hence the non-Abelian propagator  $\tilde{K} = \text{diag}(K_\alpha, K_{-\alpha})$  is simply

$$(11) \quad \tilde{K} = \tilde{C}_\alpha \sum_{n=-\infty}^{\infty} K_n^0 \Phi^n$$

confirming once more the predictions of Wu and Yang.

$\tilde{C}_\alpha = \text{diag}(C_\alpha, C_{-\alpha})$  is again an unobservable  $SU(2)$ -valued phase factor. The explicit formula (9) allows also for rederiving the scattering-expression in [6]. Observe finally that, due to the covariant transformation property  $\tilde{\Phi} \rightarrow \tilde{g}^{-1} \tilde{\Phi} g$ , (11) is valid in any gauge.

Notice that it is the existence of the diagonal gauge (6) which makes it unnecessary to working with non-commuting path integrals. However, this is true over an arbitrary Riemann surface, not only for the punctured plane [9].

### References

- [1] Wu T.T. and Yang C.N., Phys. Rev. D12, 3845 (1975)
- [2] Aharonov Y. and Bohm D., Phys. Rev 115, 485 (1959)
- [3] Botelho L. and De Mello, Rio de Janeiro preprint  
CBPF - NF - 062185 (1985)
- [4] Byers N. and Yang C.N., Phys. Rev. Lett. 7, 46 (1961)
- [5] Goddard P. and Olive D., Rep. Progr. Phys. 41, 1357 (1978)  
see also Halpern M.B., Nucl. Phys. B139, 477 (1978)
- [6] Horváthy P.A., Phys. Rev. D33, 407 (1986); see also Asorey M.,  
J. Math. Phys. 22, 179 (1981)
- [7] The importance of boundary conditions is explained in Schulman L.S.,  
Techniques and applications of Path Integration, New York, Wiley (1981):  
see also Henneberger W.C., Phys. Rev. Lett. 52, 573 (1984).
- [8] Morandi G. and Menossi E. Eur. J. Phys. 5, 49 (1984); Berry M.V.,  
Eur. J. Phys. 1, 240 (1980)
- [9] Friedrich Th and Haberman L., Comm. Math. Phys. 100, 231 (1985)