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Symmetry Restoration of Higgs Models at Finite Temperature

Yasushi Fujimoto[1]

Institut für theoretische Physik, Freie Universität Berlin
D-1000 Berlin 33

Andreas Wipf[2] and Hiroshi Yoneyama[3]

Dublin Institute for Advanced Studies
10 Burlington Road, Dublin 4, Ireland

The symmetry restoration of Higgs models at finite temperature and in less than 4 dimensions is investigated. For that purpose a series of approximations to the constraint effective lattice potential is introduced. The continuum limit of these mean-field like effective potentials is discussed and it is shown that the symmetry is always restored at finite temperature. As an application we derive an estimate for the critical temperature.

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[3] present address: Institut fur theoretische Physik, RWTH Aachen

1 Introduction

A widely used method to study the combined quantum- and temperature effects in a (continuum or lattice) quantum field theory is to use the finite temperature ($T > 0$) effective potential. The generalization from $T = 0$ to $T > 0$ proves to be very simple since the same one-particle irreducible vacuum diagrams which generate the zero-temperature potential also generate the finite-temperature potential [1]. The T-dependence of the effective potential comes solely from the boundary conditions. For $T > 0$ one imposes periodic boundary conditions for imaginary time which results in a T-dependent free propagator. This explains why the diagrammatic method, shortly after it had been developed, was employed to study temperature effects [2,3]. In this way several authors derived a perturbative value for the critical temperatures of both scalar- and gauge theories.

However, the applied loop expansion is plagued with infrared divergences which bars one from predicting a reliable value for the critical temperature in less than 4 space-time dimensions. Even worse, it predicts that certain Higgs models with two scalar fields show no symmetry restoration [2], contrary to our expectation and to the improved method of summing over all daisy graphs [4].

Alternatively to the conventional perturbative treatment one may employ the lattice formulation to explore the phase diagrams of finite temperature systems. This can be done, for example, by using numerical methods (e.g. MC) in order to simulate these lattice systems. However, there are few *quantitative* analytical lattice results for finite temperature scalar models. One reason being that the ordinary mean field (MF) approximation, which is such a useful tool to investigate the phase structure at zero $T = 0$ [5], is volume independent and thus temperature independent. To incorporate the temperature effects one must keep the finite size effects, at least in the imaginary time direction. Finally, one also must let the lattice spacing approach zero, otherwise the system may pretend properties which are artifacts of the lattice regularization. For example, there exist models on a lattice with fixed lattice constant which remain broken at all temperatures, contrary to the corresponding continuum models.

The development of the paper is as follows: in sect.2 the relevant properties of the finite temperature effective potentials are recalled. We also show that ϕ^4-theories on lattices with fixed lattice constant may be broken at *all* 'temperatures'. Mean field like potentials, which incorporate the temperature effects, are introduced and discussed in sect.3. In sect.4 we find the renormalization flow in these MF-like theories by reinterpreting the lattice constant as $\hbar$, which allows the application of the ordinary loop expansion. Then we use the obtained flow in order to discuss the continuum limit of the MF-like effective potentials. The last section is devoted to the problem of symmetry restoration of non-symmetric 2 component models which show no symmetry restoration in the conventional loop expansion. We find that in 3 dimensions these models find themselves always in the symmetric phase at sufficiently high temperature and we give an estimate for the critical temperature.

2 Lattice Potentials at Finite Temperature

Consider a field theory described by a lagrangian density $L(\phi(x))$, where $\phi(x)$ is a Higgs field on Ω . To study the combined quantum- and finite temperature corrections to the classical potential $V(\phi)$ in the classical action

$$S[\phi] = \int_\Omega L(\phi(x))\, d^dx = \int_\Omega \{\frac{1}{2}(\nabla\phi(x))^2 + V(\phi(x))\}\, d^dx \qquad (2.1)$$

one introduces effective potentials (EP). Most approaches to this subject begin with the Schwinger function

$$W(\beta, V, j) = \frac{1}{\Omega}\, ln \int D\phi\, e^{-S[\phi] + j\int_\Omega \phi(x) d^dx} \qquad (2.2)$$

in the presence of a constant external current j. The temperature dependence of $W(\beta, V, j)$, where $\Omega = \beta \times V = \beta \times L^{d-1}$ and β denotes the inverse temperature, is hidden in the boundary conditions on the allowed configurations $\phi(x) = \phi(\tau, \vec{x})$ in (2.2): At a given temperature T one sums over fields which are periodic in the (imaginary) time τ with period $\beta = 1/kT$.

In the ordinary approach one defines the *conventional effective potential* $\Gamma(\beta, V, \Phi)$ as the Legendre transform of $W(\beta, V, j)$, namely

$$\Gamma(\beta, V, \Phi) = \sup_j \{j\Phi - W(\beta, V, j)\} = (L\,W)(\beta, V, \Phi) \qquad (2.3)$$

to discuss the temperature dependence of the theory. Alternatively one may use the *constraint effective potential*

$$U(\beta, V, \Phi) = -\frac{1}{\Omega} \ln \int D\phi\ \delta\Big(\frac{1}{\Omega}\int \phi(x) d^d x - \Phi\Big)\ e^{-S[\phi]} \qquad (2.4)$$

which has been introduced by Fukuda and Kyriakopolous [6]. It is known [7] that at $T = 0$ these two potentials approach each other when the volume tends to infinite. This zero temperature result generalizes immediately to the finite temperature case.

For finite volumes it follows from (2.2) and (2.4) that

$$e^{\Omega W(\beta,V,j)} = \int d\Phi\ e^{\Omega\{j\Phi - U(\beta,V,\Phi)\}} \qquad (2.5)$$

and thus W and Γ can always be recovered from U (and conversely). As has been shown in [7], the constraint EP $U(\beta, V, \Phi)$ is not necessarily convex for finite volumes. Only in the limit $L \to \infty$ it coincides with the convex EP $\Gamma(\beta, V, \phi)$.

One way to regularize the above formal expressions is to discretize the space-time region Ω by a d-dimensional lattice with lattice spacing a. By rescaling the field, current, masses and coupling constants according to their dimensions, e.g. $\phi_i = a^{d/2-1}\phi(x_i)$, $j_i = a^{d/2+1}j(x_i)$, $m_L = a^2 m$ etc. $(i = 1, 2, ..., \Lambda = l \times N^{d-1} = \beta/a \times (L/a)^{d-1})$ the lattice action reads

$$S[\phi] = \frac{1}{2}\sum_{NN}(\phi_i - \phi_j)^2 + \sum_i V(\phi_i). \qquad (2.6)$$

The first sum is over all nearest neighbour pairs and $V(\phi_i)$ is the classical potential with rescaled parameters. As explained we must impose periodic boundary conditions in the time direction, while in the remaining spatial directions we, for convenience, assume periodicity as well. It follows that the regularized Schwinger function and effective potentials are (up to a factor a^{-d} and a-dependent constants) equal to their lattice counterparts

$$W(l, N^{d-1}, j) = \frac{1}{\Lambda} \ln \int \prod d\phi_i \, e^{j \sum \phi_i - S[\phi]} \tag{2.7}$$

and

$$U(l, N^{d-1}, \Phi) = -\frac{1}{\Lambda} \ln \int \prod d\phi_i \, \delta(\frac{1}{\Lambda} \sum \phi_i - \Phi) \, e^{-S[\phi]}. \tag{2.8}$$

In what follows we shall be interested in the dependence of the various potentials on the 'temperature' l^{-1}, the spatial 'volume' N^{d-1} and the dimension d. So we keep these quantities explicitly in the Schwinger function and the constraint EP. We have used quotation marks to indicate that l and N^{d-1} are both dimensionless, unphysical numbers. They are related to their physical counterparts by a dimensionful scale parameter.

For a fixed lattice constant a one lets the spatial 'length' N of the lattice approach infinity. We assume that the Higgs model is spontaneously broken for large l (low 'temperature'), i.e. that for sufficiently large l's the expectation value of the Higgs field

$$\langle \phi_i \rangle^l = \lim_{j \searrow 0} \frac{d\, W(l, \infty, j)}{d\, j}, \tag{2.9}$$

is strictly positive, or equivalently that the Schwinger function develops a singularity when the spatial 'volume' tends to ∞, i.e.

$$W''(l, N^{d-1}, j = 0) = \sum_i \langle \phi_0 \phi_i \rangle^l, \tag{2.10}$$

approaches infinity when $N \to \infty$. To see whether the model exhibits a symmetry restoration, i.e. has a critical 'temperature' l_c^{-1} such that $\langle \phi_i \rangle^l > 0$ for $l > l_c$ and $\langle \phi_i \rangle^l = 0$ for $l < l_c$, one squeezes the lattice in the time direction. When the Schwinger function becomes smooth for small l (high 'temperature'), then the symmetry is restored. In what follows we take for simplicity a ϕ^4-model and choose the parametrization $V(\phi) = m\phi^2 + g\phi^4$ for the classical lattice potential. We furthermore keep the dependence of the Schwinger function on the mass m.

Then, the main result of this section is as follows:
In 3 or more dimensions and for sufficiently negative masses m the symmetry remains broken at all 'temperatures' l^{-1}.

In what follows we shall see that this result follows from the inequality

$$W''(l, N^{d-1}, m, j = 0) \geq W''(N, N^{d-2}, m + 1, j = 0), \qquad (2.11)$$

which holds for any l and any N by letting N tend to infinity. The crucial inequality (2.11) compares the curvatures of the Schwinger functions of two different models: the d-dimensional finite 'temperature' model on a $l \times N^{d-1}$ lattice with the $(d-1)$-dimensional zero 'temperature' model on a $N \times N^{d-2} = N^{d-1}$ lattice, but with a shifted mass. The point is that zero 'temperature' ϕ^4-models have been extensively studied and are known to be spontaneously broken when $d \geq 2$ and $m < m_c < 0$ [8], in which cases their Schwinger functions develop a singularity in the thermodynamic limit. This, together with the inequality (2.11), tells us that

$$\lim_{N \to \infty} W''(l, N^{d-1}, m, j = 0) = \infty$$

when $d \geq 3$ and $m < m_c - 1$, irrespective of the 'temperature' l^{-1}. To sum up, therefore, for a fixed lattice constant a or equivalently for fixed bare parameters (m,g), the squeezing of the lattice does not necessarily force the system into a symmetric state.

To derive (2.11) we consider the one-parametric family of actions

$$S_\epsilon[\phi] = -\sum_{sl\ NN} \phi_i \phi_j \; - \epsilon \sum_{tl\ NN} \phi_i \phi_j + \sum_i V_d(\phi_i) \qquad (2.12)$$

which interpolates between l copies of a $(d-1)$-dimensional model and the original theory (2.6). The first (second) sum in (2.12) is over

ike) nearest neighbour pairs and $V_d(\phi) = d\,\phi^2 + V(\phi)$.
ths inequality one can show (see appendix A) that for
any finite N

$$\frac{d}{d\epsilon}\,\langle\phi_0\phi_i\rangle^l_\epsilon \geq 0, \qquad (2.13)$$

denotes the expectation value of $O[\phi]$ computed with

$$\langle O[\phi]\rangle_\epsilon = \frac{\int\prod d\phi_i\, O[\phi]\, e^{-S_\epsilon[\phi]}}{\int\prod d\phi_i\, e^{-S_\epsilon[\phi]}}. \qquad (2.14)$$

ired inequality (2.11) we integrate the 'differential in-
) from $\epsilon = 0$ to $\epsilon = 1$ and sum over i. For $\epsilon = 1$ the
ecomes (2.6) and the sum $\sum_i \langle\phi_0\phi_i\rangle^l_{\epsilon=1}$ is, according
left hand side of the inequality (2.11). In order to see
r the right hand side of (2.11) when $\epsilon = 0$ we observe
se the action (2.12) belongs to l non-interacting copies
mensional model. Thus $\langle\phi_0\phi_i\rangle^l_{\epsilon=0}$ vanishes if the sites 0
ferent time slices of the lattice. When they belong to the
n the integrals over fields on the other $l-1$ slices cancel
n (2.14). One sees at once that the remaining $(d-1)$-
ction (on the slice defined by ϕ_0 and ϕ_i) has a shifted
+ 1, and therefore the sum $\sum_i\langle\phi_0\phi_i\rangle^l_{\epsilon=0}$ coincides with
d side of the inequality (2.11).

t sight this shift of the bare mass may look insignificant.
bare parameters have to be related to physical quantities
ation and we have to consider whether this renormaliza-
ır conclusions. Indeed in less than 4 dimensions the bare
) zero when a tends to zero (see sections 4 and 5). Thus,
e 'effective mass' m+1 in the right hand side of (2.11) be-
e and $W"(N, N^{d-2}, m+1, j=0)$ stays finite as $N \to \infty$.
uum limit the inequality (2.11) is therefore of no value
ie symmetry restoration at high temperature.
on to be drawn from our analysis is that it is essential to
inuum limit in studying the finite temperature behaviour
ulated field theories. The squeezing of the lattic alone does
e the symmetry restoration.

3 Mean Field like Potentials

The most crude one-body approximation, i.e. mean field (MF) theory, provides us with a good qualitative picture of the phase structure at zero temperature [5]. It serves as an important ingredient in our understanding of lattice theories. However, in the ordinary MF- approximation one loses the volume dependence of the lattice potentials and hence the temperature dependence of the theory. In this section we formulate a modified mean field approximation to the constraint effective potential which incorporates the finite temperature effects.

3.1 *Approximations to the Constraint Effective Potential*

More generally we introduce a series of approximations, labelled by an integer $0 \leq p \leq d$, such that for p=0 we recover the exact theory and for p=d the ordinary MF approximation. Let us, for this purpose, consider the d-dimensional lattice Λ as product $\Lambda = \Lambda_p \times \Lambda_q$, (q=d-p). We indicate this decomposition through the notation $\phi_{i,j}$ for the field at the site $(i,j) \in \Lambda_p \times \Lambda_q$. For example, when $p = d-1$ and therefore $q = 1$ then Λ_{d-1} may be thought of as a time slice of the d-dimensional lattice and Λ_1 as the sites with the same spatial coordinates.

Instead of replacing the interaction of $\phi_{i,j}$ with its nearest neighbours by the interaction with the mean field $M = \frac{1}{\Lambda}\sum_{\Lambda} \phi_{k,l}$ due to all site variables, as it is done in the ordinary MF approximation, we make this approximation only for nearest neighbours with the same Λ_q-coordinates. More explicitly, we replace the action (2.6) in the exponent in (2.8) by $\frac{1}{2}\sum_{NN}(\phi_{i,j} - \phi_{i,l})^2 - p\Lambda M^2 + \sum V_p(\phi_{i,j})$, where $V_p(\phi) = p\phi^2 + V(\phi)$. For the following manipulations it is essential that the first sum is only over neighbours with the same Λ_p coordinates. Note also that, due to the constraint in (2.8), the second term becomes $-p\Lambda\Phi^2$. Next we insert the identity

$$\delta\Big(\frac{1}{\Lambda}\sum_{\substack{i,j\in\\ \Lambda_p\times\Lambda_q}} \phi_{i,j} - \Phi\Big) = \int \prod_{i=1}^{\Lambda_p} d\Phi_i \; \delta\Big(\frac{1}{\Lambda_q}\sum_j \phi_{i,j} - \Phi_i\Big)\, \delta\Big(\frac{1}{\Lambda_p}\sum_i \Phi_i - \Phi\Big)$$

for the constraint. By observing that the Boltzmann factor in (2.8) factorizes, one obtains, after integation over the fields $\phi_{i,j}$, the approximating potential

$$U_p(\Lambda_q,\Phi) = -p\Phi^2 - \frac{1}{\Lambda}\ln\int\prod_i d\Phi_i\,\delta(\frac{1}{\Lambda_p}\sum\Phi_i - \Phi)\;e^{-\Lambda_q U(m+p,\Phi_i)},$$

where $U(m+p,\Phi_i)$ denotes the constraint EP for a q-dimensional lattice theory on Λ_q with a shifted mass $m\to m+p$. Next we insert the Fourier representation

$$\delta(\frac{1}{\Lambda_p}\sum\Phi_i - \Phi) = \Lambda\int dk\; e^{i\Lambda k\Phi - ik\Lambda_q\sum\Phi_i}$$

of the δ-distribution and end up with

$$U_p(\Lambda_q,\Phi) = -p\Phi^2 - \frac{1}{\Lambda}\ln\big(\Lambda\int dk\; e^{\Lambda\{ik\Phi + W(m+p,-ik)\}}\big) \qquad (3.1)$$

where $\exp\{\Lambda_q W(m+p,-ik)\} = \int d\Phi\,\exp\{-\Lambda_q(ik\Phi + U(m+p,\Phi))\}$. In the limit when the volume tends to infinity the integral (3.1) coincides with its value at the saddlepoint on the imaginary axis. Thus we obtain $(k = i\cdot j)$

$$U_p(\Lambda_q,\Phi) = -p\Phi^2 + \sup_j\{j\Phi - W(m+p,j)\} \qquad (3.2)$$

where

$$\begin{aligned} W(m+p,j) &= \frac{1}{\Lambda_q}\ln\int e^{\Lambda_q(j\Phi - U(m+p,\Phi))}\,d\Phi \\ &= \frac{1}{\Lambda_q}\ln\int\prod_{\Lambda_q} d\phi_j\; e^{j\sum\phi_j - S[m+p,\phi]} \end{aligned} \qquad (3.3)$$

is the Schwinger function of a q=(d-p) dimensional lattice Higgs model which differs from the original theory by a shifted mass. With (3.2) we finally derived the desired approximations to the exact constraint EP (2.8). Note that U_p belongs to a $q = d-p$ dimensional reparametrized lattice model $(m\to m+p)$. Still we are left with the functional integral (3.3) on the lattice Λ_q. Clearly with increasing p the MF-like potentials (3.2) become better approximations to the exact constraint EP. Especially for p=0 one recovers the exact potential.

3.2 The Case $p = d$ (Ordinary MF Approximation) and Symmetry Restoration

In the other extreme case, p=d, $W(m + d, j)$ is the Schwinger function of a zero-dimensional 'field theory' with mass m+d. In other words,

$$U_d(\Phi) = -d\Phi^2 + (LW)(m + d, \Phi) = -d\Phi^2 + \Gamma(m + d, \Phi) \qquad (3.4)$$

where

$$W(m + d, j) = \ln \int d\phi \, e^{j\phi - V_d(\phi)} \qquad (3.5)$$

is given by an ordinary integral and is independent of Λ. Such zero-dimensional models with a shifted mass, so-called incoherent models, were used in [7] to bound the constraint EP from below and above.

At a minimum Φ_0 of the effective potential $U_d'(\Phi_0) = -2d\Phi_0 + j(\Phi_0) = 0$. Here $j(\Phi) = \Gamma'(m + d, \Phi)$ is the current which belongs to Φ. Since Γ is the Legendre transform of W, the inverse relation reads $\Phi(j) = W'(m+d, j)$. By inserting the minimum condition into the last equation we end up with the well-known self consistency equation [5]

$$\Phi_0 = \frac{\int d\phi \, \phi \, e^{2d\Phi_0\phi - V_d(\phi)}}{\int \, d\phi \, e^{2d\Phi_0\phi - V_d(\phi)}} = \langle\phi\rangle_{2d\Phi_0} \qquad (3.6)$$

for the MF expectation value of the Higgs field. So one recovers the ordinary mean field approximation for p=d.

Later on we will use the curvature of the constraint EP at its minima to renormalize the parameters of the potential. For its computation one uses the relation $\Gamma"(\Phi) = W"(j(\Phi))^{-1}$ between the curvatures of Γ and W. Together with the minimum condition $j(\Phi_0) = 2d\Phi_0$ one obtains

$$m_0 = U_d"(\Phi_0) = -2d + \langle(\phi - \Phi_0)^2\rangle_{2d\Phi_0}^{-1} \qquad (3.7)$$

for the Higgs mass in the broken phase.

Clearly the incoherent Schwinger function in (3.5) is strictly convex and symmetric and hence $j(\Phi=0)=0$. Together with $W''(0)=\langle\phi^2\rangle_0$, we conclude that the curvature of U_d at the origin, $U_d''(0)=-2d+W''(m+d,0)^{-1}$, is negative when $\langle\phi^2\rangle_0 > 1/2d$. Consequently the MF potential (3.4) is spontaneously broken in cases where

$$\frac{\int \phi^2\, e^{-V_d(\phi)}}{\int e^{-V_d(\phi)}} = \langle\phi^2\rangle_0 > 1/2d. \tag{3.8}$$

Suppose, for example, that $m \le -d$. Since the m-derivative of $\langle\phi^2\rangle_0$, $d\langle\phi^2\rangle_0/dm = -\langle(\phi^2-\langle\phi\rangle_0^2)^2\rangle_0$ is negative, the expectation value $\langle\phi^2\rangle_0$ decreases with increasing mass and becomes smaller when m is replaced by -d. However, for this value the effective mass (m+d) vanishes and $\langle\phi^2\rangle_0$ can be computed explicitly. In this way one finds that the MF-potential is spontaneously broken for $m\le -d$ and $g < \{2d\,\Gamma(3/4)/\Gamma(1/4)\}^2$. These results can easily be generalized to the case where the Higgs field has several components: Consider for simplicity an even potential $V(\phi_1^2,\dots,\phi_n^2)$. Then the matrix $\partial_a\partial_b W(m+d,0)$ and its inverse $\partial_a\partial_b\Gamma(m+d,0)$ are both diagonal. One sees at once that the condition (3.8) for a spontaneous symmetry breakdown is now replaced by

$$\max\{\langle\phi_1^2\rangle_0,\dots,\langle\phi_n^2\rangle_0\} > 1/2d. \tag{3.8$'$}$$

In fig.1 the MF-approximation (3.4) to the constraint EP is compared with the results of Monte Carlo simulations on a 8-dimensional and a 4-dimensional lattice with 160 and 8^4 lattice sites respectively. For the chosen parameters the approximation is surprisingly accurate.

Figure 1

3.3 *The Case* $p = d - 1$ *(Modified MF-approximation)*

As pointed out earlier, the potential $U_d(\Phi)$ is independent of the volume. However, at finite temperature we must keep the finite size effects due to a varying lattice-length in the time direction. This suggests that we keep the NN-interactions in the time direction and approximate in the remaining spatial direction(s). So we take $p = d - 1$ and $\Lambda_q = \Lambda_1 = l$ in (3.2) and call

$$U_{d-1}(l, \Phi) = -(d-1)\Phi^2 + \Gamma(m + d - 1, \Phi) \qquad (3.9)$$

the *modified MF effective potential.* Note, that now Γ is the Legendre transform of a quantum mechanical Schwinger function

$$W(m + d - 1, j) = \frac{1}{l} \ln \int \prod_1^l d\phi_i \; e^{j \sum \phi_j - S[m+d-1, \phi]}, \qquad (3.10)$$

and we are left with a one-dimensional field theory with a shifted mass $m \to m + d - 1$. One sees at once that the generalizations of the self consistency equation (3.6) and the Higgs mass (3.7) read

$$\Phi_0 = \langle M \rangle_{2(d-1)\Phi_0} \qquad (3.11)$$

$$m_0 = U''_{d-1}(\Phi_0) = -2(d-1) + \langle (M - \Phi_0)^2 \rangle^{-1}_{2(d-1)\Phi_0}, \qquad (3.12)$$

where the expectation values of $M = \frac{1}{l} \sum \phi_i$ and $(M - \Phi_0)^2$ are computed with the integrand in the right side of (3.10) and j is replaced by $2(d-1)\Phi_0$.

By applying the inequality (2.11), namely $\langle M^2 \rangle_0 \geq \langle \phi^2 \rangle_0$, where the second expectation value is defined in (3.8), one obtains the upper bound $U''_{d-1}(l, 0) \leq -2(d-1) + \langle \phi^2 \rangle_0^{-1}$ for the curvature of U_{d-1} at the origin. Thus, the modified MF effective potential (3.9) is spontaneously broken at all temperatures l, when

$$\frac{\int \phi^2 \, e^{-V_d(\phi)}}{\int e^{-V_d(\phi)}} > \frac{1}{2(d-1)}. \qquad (3.13)$$

This implies that there will never be a symmetry restoration at finite temperature if, for example, $m < -(d-1)$ and $g < \{2(d-1)\Gamma(3/4)/\Gamma(1/4)\}^2$, similarly to the statement below (3.8). As we have seen in the last section, this is not a peculiarity of the mean field approximation. In the full lattice theory one has the same no-go theorem. Once more we conclude that it is essential to take the continuum limit in order to study the temperature dependence of U_{d-1}.

Before turning to this limit let us add some more remarks: Although we discussed the special cases $p = d$ and $p = d-1$ of the series of approximations (3.2) to the constraint EP, it should be obvious how these results apply to the other cases. Furthermore, it is clear that whenever the self consistency equation allows non trivial solutions, then the MF-like potentials $U_p(\Lambda_q, \Phi)$ are non-convex. This is true even in the infinite volume limit, when the exact potential becomes convex.

Finally, we wish to point out, that the potentials (3.2) can be regarded as approximations to the conventional EP rather than the constraint EP. To see that, one uses the (Gibbs) variational characterization of the conventional EP

$$\Gamma(\Lambda, \Phi) = \min_{\substack{\Psi[\phi_1,\dots,\phi_\Lambda] \\ \int \phi_i \Psi[\phi] = \Phi \\ \int \Psi[\phi]=1}} \frac{1}{\Lambda} \int \prod d\phi_i \, \{\Psi[\phi]\,(S[\phi] + \ln \Psi[\phi])\} \tag{3.14}$$

which easily can be proven by standard Euler-Lagrange methods for constraint systems [9]. It turns out that the right side becomes the approximating potential $U_p(\Lambda_q, \Phi)$ in (3.2) if one minimizes only with respect to product densities of the form

$$\Psi[\phi] = \prod_{i \in \Lambda_p} f(\phi_{i,1}, \dots, \phi_{i,\Lambda_q}),$$

where, as earlier, $\phi_{i,j}$ denotes the site variable at site $(i,j) \in \Lambda_p \times \Lambda_q$. For example, one recovers the modified MF effective potential (3.9) by assuming that Ψ factorizes in all spatial directions and by keeping only the NN-interactions in the time direction.

4 The Continuum Limit

In the preceding sections we have not introduced any explicit renormalization. However, the bare quantities (m,g) we have considered have to be related to physical quantities by renormalization. As *physical parameters* we take the expectation value of the Higgs field Φ_p and the Higgs mass m_p in the broken phase. One conveniently introduces a dimensionless lattice constant $\lambda = a \cdot \mu$, where μ is a scale parameter of mass dimension, and measures the various physical quantities in μ-units.

Let us first consider the *ordinary* MF potential (3.4) on Z^d. To construct the scaling limit one compares the lattices Z^d and $(\lambda Z)^d$ when λ is allowed to take values in the interval $0 < \lambda < 1$. One sees at once that the potential on $(\lambda Z)^d$ becomes

$$\begin{aligned} U_d^\lambda(\Phi) &= \lambda^{-d} U_d(\lambda^{\frac{d}{2}-1}\Phi) \\ &= -d\lambda^{-2}\Phi^2 + \lambda^{-d}\Gamma(m(\lambda)+d,\, g(\lambda)\,;\lambda^{\frac{d}{2}-1}\Phi), \end{aligned} \tag{4.1}$$

where the scaled bare parameters $m(\lambda)$ and $g(\lambda)$ are to be determined by some renormalization condition [9]. As fixed physical parameters we take the expectation value Φ_p which minimizes U_d^λ and the Higgs mass $m_p = U_d^{\lambda}{}''(\Phi_p)$.

Obviously, when Φ_p minimizes U_d^λ then $\lambda^{\frac{d}{2}-1}\Phi_p$ minimizes U_d and fulfils the selfconsistency equation (3.6). Thus, the first renormalization condition reads

$$\lambda^{\frac{d}{2}-1}\Phi_p = \langle\phi\rangle_{2d\lambda^{\frac{d}{2}-1}\Phi_p}, \tag{4.2}$$

where the expectation value was defined in (3.6). Analogously, by using (3.7) one obtains the second renormalization condition

$$\lambda^2 m_p = \lambda^2 U_d^{\lambda}{}''(\Phi_p) = -2d + \langle(\phi - \lambda^{\frac{d}{2}-1}\Phi_p)^2\rangle^{-1}_{2d\lambda^{\frac{d}{2}-1}\Phi_p}. \tag{4.3}$$

To find the asymptotic form of the bare parameters in the classical potential $V = m\phi^2 + g\phi^4$ for small λ one expands the expectation values in (4.2) and (4.3) around $\lambda = 0$. Since the leading λ behaviour of m and g is not known a priori, we used the solvable model $V(\phi) = (m+\sqrt{2g})\phi^2 - \ln(1+\sqrt{2g}\phi^2) \sim m\phi^2 + g\phi^4 + \ldots$ for making a first guess. Next, by inserting the small-λ expansion into the above renormalization conditions we determined the coefficients of λ^n and found the following renormalization flows in 2 and 3 dimensions:

$$\begin{aligned} d=2: \qquad & g(\lambda) \sim \frac{m_p}{8\Phi_p^2}\lambda^2 \qquad & m(\lambda) \sim -(\frac{3}{2} + 2\Phi_p^2)g(\lambda), \\ d=3: \qquad & g(\lambda) \sim \frac{m_p}{8\Phi_p^2}\lambda \qquad & m(\lambda) \sim -g(\lambda). \end{aligned} \tag{4.4}$$

In 4 dimensions both, $m(\lambda)$ and $g(\lambda)$, approach a constant value when $\lambda \to 0$. For an alternative method to derive (4.4) one can employ similar arguments like those presented in the appendix B. In order to find the 'exact' flow we solved the equations (4.2) and (4.3) numerically for $\lambda = 1, 2^{-1}, \ldots, 2^{-8}$. In fig.2 the numerical results are compared with the the asymptotic flow (4.4) in 3 dimensions. One sees that the ratios approach 1 as λ tends to 0 rather quickly.

Figure 2

Instead of fixing the expectation value of the field and the Higgs mass, one also could use $2\alpha_p = U_d^{\lambda\prime\prime}(0)$ and $4!\beta_p = (U_d^{\lambda})^{IV}(0)$ as physical parameters. The corresponding renormalization conditions

$$\langle\phi^2\rangle_0 = \omega \qquad \text{and} \qquad \langle\phi^4\rangle_0 = \theta, \tag{4.5}$$

where $2\omega = (d+\lambda^2\alpha_p)^{-1}$ and $\theta = 2\omega^2(1-8\lambda^{4-d}\beta_p\omega^2)$, are inconsistent when $\beta_p > 0$ and $d > 4$, because ω approaches the finite value 1/2d when λ tends to zero and therefore θ becomes negative. However, the second equation in (4.5) demands θ to be positive and so β_p must vanish in more than 4 dimensions. Since for $\beta_p = 0$ the connected 4-point function vanishes, the theory becomes trivial (similar arguments show that the higher connected n-point functions vanish as well).

Let us now consider the scaling limit of the *modified* MF-potential (3.9). For that purpose one compares the lattices $l \times Z^{d-1}$ and $(\lambda l) \times (\lambda Z)^{d-1}$, where the (dimensionless) inverse temperature $\beta = \lambda l$ is to be kept fixed. To determine the renormalization flow of the bare parameters in the scaled potential

$$U^{\lambda}_{d-1}(\beta, \Phi) = -(d-1)\lambda^{-2}\Phi^2 + \lambda^{-d}\,\Gamma(m(\lambda)+d-1,\, g(\lambda)\,, \lambda^{\frac{d}{2}-1}\Phi) \quad (4.6)$$

we, like in the $p = d$ MF-approximation, fix the expectation value Φ_p and the Higgs mass m_p. Since it suffices to renormalize the bare parameters at zero temperature (l=∞) in order to regularize the finite temperature ($l < \infty$) theory, we compute the renormalization flow on the lattice $(\lambda Z) \times (\lambda Z)^{d-1}$, by using the renormalization conditions

$$\lambda^{\frac{d}{2}-1}\Phi_p = \langle M \rangle_{2(d-1)\lambda^{\frac{d}{2}-1}\Phi_p} \quad (4.7)$$

and

$$\begin{aligned} \lambda^2 m_p &= \lambda^2 U^{\lambda}_{d-1}{}''(\infty, \Phi_p) \\ &= -2(d-1) + \langle (M - \lambda^{\frac{d}{2}-1}\Phi_p)^2 \rangle^{-1}_{2(d-1)\lambda^{\frac{d}{2}-1}\Phi_p}, \end{aligned} \quad (4.8)$$

where the expectation values are computed with the integrand in (3.10), wherein $l = \infty$ and j is replaced by $2(d-1)\lambda^{\frac{d}{2}-1}\Phi_p$.

To proceed we must approximate or alternatively simulate the quantum mechanical EP Γ on the right hand of (4.6). To find a sensible approximation to Γ one observes that, since $m(\lambda)$ tends to zero in the continuum limit, the effective mass $m(\lambda) + d - 1$ is positive for small λ and thus the ordinary loop expansion is not plagued with infra-red divergences, even when the original theory is spontaneously broken. Later on we will see that the λ-dependence of the parameters and the field does not ruin the expansion

$$\Gamma(m + d - 1, g, \lambda^{\frac{d}{2}-1}\Phi) = V_{d-1}(.,.,.) + V_{1-loop}(.,.,.) + \ldots. \quad (4.9)$$

In fact, it turns out that V_{r-loop} is of order $O(g^{r-1})$ relative to the 1-loop contribution, and since the bare coupling constant tends to zero in the continuum limit, the 1-loop correction (at zero temperature) gives the *correct* asymptotic renormalization flow. So it suffices to take the quantum mechanical 1-loop contribution on the lattice $V_{1-loop}(\Phi) = \frac{1}{2}\, arch(1 + \frac{1}{2}V^{"}_{d-1}(\Phi))$ [10]. Part of the classical mass term in (4.9) cancels the first term on the right side in (4.6) and we end up with the expansion

$$U^{\lambda}_{d-1}(\infty, \Phi) \sim m\lambda^{-2}\Phi^2 + g\lambda^{d-4}\Phi^4 + \frac{\lambda^{-d}}{2} arch\,\{1 + (m+d-1)(1+\delta)\} \tag{4.10}$$

where $\delta = 6g\lambda^{d-2}\Phi^2/(m+d-1)$. After expanding the 1-loop contribution in powers of δ and by using the renormalization conditions (4.7) and (4.8), one obtains the following asympotic renormalization flows in 2 and 3 dimensions

$$\begin{aligned} d=2: &\qquad g(\lambda) \sim \frac{m_p}{8\Phi_p^2}\lambda^2 \qquad m(\lambda) \sim -(\sqrt{3} + 2\Phi_p^2)g(\lambda) \\ d=3: &\qquad g(\lambda) \sim \frac{m_p}{8\Phi_p^2}\lambda \qquad m(\lambda) \sim -\frac{3}{2\sqrt{2}}g(\lambda). \end{aligned} \tag{4.11}$$

By comparing (4.11) with (4.4) one sees that the ordinary MF- and the modified MF-approximation give rise to the same λ- dependence of the bare parameters in 2 and 3 dimensions. This is not very surprising, since U_d and U_{d-1} only differ by a MF-approximation in the (one dimensional) time direction and since a one-dimensional field theory needs no renormalization.

Let us now check that the λ-dependence of the parameters and the field in (4.9) does not spoil the above expansion of the effective potential. For that we apply the results in [11], and find the general form for the r-loop contribution to the effective potential

$$V_{r-loop} = \sqrt{m+d-1}\,\Big(\frac{g}{(m+d-1)^{3/2}}\Big)^{r-1} F_r(\Delta),$$

where $\Delta = 1+\delta$ is dimensionless and F_r has the expansion $F_r(\Delta) \sim a + c\delta + c\delta^2 + \ldots$. Thus, up to a constant, the r-loop contribution in (4.9),

$$V_{r-loop}(.,.,.) \sim \lambda^{-2} g^r (c_1\phi^2 + c_2\lambda^{d-2}g\phi^4 + \ldots),$$

is of order $O(g^{r-1})$ relative to the 1-loop contribution. It follows that, since the bare coupling constant g approaches zero in the continuum limit, one may neglect the higher loop contributions to the flows (4.11).

We are now ready to show that the symmetry restoration takes place in 3-dimensional Higgs models. To see this, we compute the second derivative of $U^{\lambda}_{d-1}(\beta,\Phi)$ at the origin on a lattice with 2 time-slices (l=2) that means for the highest possible temperature. For that we must calculate the curvature of the quantum mechanical Schwinger function on two sites and with scaled parameters. With (4.11) we assume the asymptotic behaviour $g(\lambda) \sim \frac{1}{8} m_p \lambda / \Phi_p^2$ and $m(\lambda) \sim -(1+\epsilon) g(\lambda)$ and find the following asymptotic form (see appendix B, where the computation for a more general model is presented)

$$U^{\lambda}_{d-1}{}''(\beta,0) \sim \alpha \, \frac{m_p}{32\Phi_p^2} \lambda^{-1},$$

where $\alpha = 9 - 12/\sqrt{2}$ is positive for the flow (4.11). With $\beta = l \cdot \lambda = 2\lambda \ll 1$ it follows that at high temperature the curvature of the modified MF potential at the origin,

$$U^{\lambda}_{d-1}{}''(\beta,0) \sim \alpha \, \frac{m_p}{16\Phi_p^2} T \qquad (4.12)$$

is positive and therefore the symmetry is restored. We may use (4.12) to obtain an estimate for the critical temperature. For that we add the high temperature contribution (4.12) to the zero temperature EP, which yields

$$U^{\lambda}_{d-1}(\beta,\Phi) \sim -\frac{m_p}{4}(1 - \frac{1}{8\Phi_p^2}\alpha T)\Phi^2 + \frac{m_p}{8\phi_p^2}\Phi^4 + ... \qquad (4.13)$$

The mass term changes sign at

$$T_c = 8\frac{\Phi_p^2}{\alpha} \qquad (4.14)$$

which serves as a first guess for the critical temperature of the one component Higgs model in 3 dimensions. To examine the quality of the estimate (4.14) we compare it with the corresponding MC result for the 3-dimensional Higgs model. From [12] we take the continuum value $T_c \sim 0.62$ for $\Phi_p \sim 0.308$. This value is approximately half of our crude estimate (4.14) which yields $T_c \sim 1.49$.

If we wish to make contact with the conventional loop expansion, then we should compare (4.13) with the high temperature expansion of the conventional one-loop effective potential in 3 dimensions, namely with

$$\begin{aligned}\Gamma_{1-loop}(\beta,\Phi) = & -\frac{m_p}{4}(1-\frac{3T}{2\pi\Phi_p^2})\Phi^2 + \frac{m_p}{8\Phi_p^2}\Phi^4 \\ & -\frac{\zeta(3)}{2\pi}T^3 - \frac{V''}{4\pi}T\log(\frac{V''}{4T^2}) + O(M^2/T).\end{aligned} \tag{4.15}$$

The remaining terms are positive integer powers of $M^2\beta^2$ times an overal factor $\beta^{-1}M^2$ and are negligible at high temperature. One observes that the high temperature expansion in 3 dimensions has a worse infrared behaviour than the corresponding expansion in 4 dimensions. The trouble is that already the leading mass correction (the logarithmic term) shows an infrared divergence. If we would discard this singular term in (4.15) then we recover (4.13) and (4.14), wherein $\alpha = 12/\pi$. This yields a lower critical temperature as the one we found with our method.

It may be worth remarking that we never met any infrared problems in the course of our derivations. The one-loop contribution in (4.10), which may become complex in the conventional loop expansion, stays real for sufficiently small λ for which $m(\lambda)$ is small and therefore $m+d-1$ is positive.

5 Symmetry Restoration of the Non-Symmetric 2 Component Model

In the last section we have introduced methods for discussing the symmetry restoration of scalar theories at finite temperature. We shall now apply the apparatus developed to the subtle and interesting case when there are two interacting fields with classical potential

$$V(\phi_1, \phi_2) = m_1\phi_1^2 + g_1\phi_1^4 + m_2\phi_2^2 + g_2\phi_2^4 - g_{12}\phi_1^2\phi_2^2. \qquad (5.1)$$

For the model to be stable we must assume $4g_1g_2 > g_2^2$. This model is interesting, since it shows no symmetry restoration at finite temperature in the conventional loop expansion when $g_1 > 2g_{12} > g_2$ [2]. On the other hand, when one selfconsistently solves the equations for the effective masses it shows a transition at some critical temperature [4].

To see whether the modified MF approximation predicts a symmetry restoration we first compute the renormalization flow for the bare parameters in the potential (5.1). We apply the same strategy as in the last section and use the 1-loop contribution $V_{1-loop}(\Phi_1, \Phi_2) = \frac{1}{2}tr\, arch(1 + \frac{1}{2}V^{"}_{d-1}(\Phi_1, \Phi_2))$ to the quantum mechanical effective potential Γ on the right side of (4.6). After separating the field independent and the field dependent contributions to $\frac{1}{2}V^{"}_{d-1}$, namely $(d-1)Id$ and

$$V^{"} = \begin{pmatrix} 6g_1\Phi_1^2 - g_{12}\Phi_2^2 & -2g_{12}\Phi_1\Phi_2 \\ -2g_{12}\Phi_1\Phi_2 & 6g_2\Phi_2^2 - g_{12}\Phi_1^2 \end{pmatrix}, \qquad (5.2)$$

one easily finds the small-λ expansion $V_{1-loop} = \frac{1}{2}(d^2-1)^{-1/2}\, tr\, V^{"} + \ldots$. Combining this 1-loop result with (4.9) and (4.6) finally yields the desired approxiamation to the scaled effective potential U^{λ}_{d-1}. At this point it is convenient to fix the physical parameters. Again we take the expecation values $(\Phi_{1p}, \Phi_{2p}) = \Phi_p$ of the two fields and the diagonal elements (m_{1p}, m_{2p}) of the second derivative of the effective potential at its minimum Φ_p. In order to restrict the number of parameters to four, we furthermore assume that $U^{"}_{d-1}(0)$ is proportional to the identity matrix. In terms of these parameters the potential may be written as

$$V(\phi) = \frac{1}{8}\sum_i \frac{m_{ip}}{\Phi_{ip}}(\phi_i^2 - \Phi_{ip}^2)^2 \; + \; \frac{1}{4}\frac{\Delta m_p}{\Delta\Phi_p^2}(\phi_1^2 - \Phi_{p1}^2)(\phi_2^2 - \Phi_{p2}^2) \quad (5.3)$$

where $\Delta m_p = m_{2p} - m_{1p}$ and $\Delta\Phi_p^2 = \Phi_{2p}^2 - \Phi_{1p}^2$. Note that the two fields decouple when $m_{1p} = m_{2p}$.

By compareing the one loop result for U^{λ}_{d-1} with the parametrization (5.3) one extracts the small-λ dependence of the bare parameters in U^{λ}_{d-1}. In this way one obtains in 3 dimensions

$$\begin{aligned} g_i &\sim \lambda \frac{m_{ip}}{8\Phi^2_{ip}} \qquad g_{12} \sim -\lambda \frac{\Delta m_p}{4\Delta\Phi^2_p} \\ m_i &\sim \frac{1}{4\sqrt{2}}(6g_i - g_{12}). \end{aligned} \tag{5.4}$$

One sees that the interaction term $-g_{12}\Phi_1^2\Phi_2^2$ in (5.1) changes the λ-dependence of the bare masses relative to the flow (4.11). Like in the 1-component case the higher loop contributions do not change the small-λ dependence of the bare parameters.

Again we can use the result (5.4) to obtain information about the high temperature behaviour of the 2-component theory. For that we, like in the last section, calculate the small-λ expansion of $U^{\lambda"}_{d-1}(\beta,0)$ on a lattice with 2 lattice sites in the time direction. In appendix B we show that for small values of λ

$$U^{\lambda"}_{d-1}(\beta,0) \sim \frac{\alpha}{96\lambda}\begin{pmatrix} 3\frac{m_{1p}}{\Phi^2_{1p}} + \frac{\Delta m_p}{\Delta\Phi^2_p} & 0 \\ 0 & 3\frac{m_{2p}}{\Phi^2_{2p}} + \frac{\Delta m_p}{\Delta\Phi^2_p} \end{pmatrix} + O(1), \tag{5.5}$$

where $\alpha = 9-12/\sqrt{2} > 0$. The stability condition for the parameters in (5.3) and the fact that the geometric mean of two positive numbers is less or equals to their arithmetic mean, tells us that m_{ip}/Φ^2_{ip} is greater than $2|\Delta m_p/\Delta\Phi^2_p|$ and therefore both entries in the matrix (5.5) are positive, regardless of the sign of $\Delta m_p/\Delta\Phi^2_p$. We see that for a lattice with 2 time slices, for which $\beta = 2\lambda$, the curvature of U^{λ}_{d-1} at the origin becomes always positive at high temperature. We conclude, and that is the main result of this section, that the modified MF approximation predicts a symmetry restoration for all 'stable' parameters, contrary to the conventional loop expansion.

To estimate the critical temperature we again add the high temperature contribution (5.5) to the zero temperature EP in order to see at which temperature the mass terms of the effective potential change signs. In this way one obtains

$$T_c = \max_i \Big\{ \frac{24}{\alpha} \Phi_{ip}^2 \, \frac{m_{2p}\Phi_{2p}^2 - m_{1p}\Phi_{1p}^2}{3m_{ip}\Delta\Phi_p^2 + \Phi_{ip}^2 \Delta m_p} \Big\} \tag{5.6}$$

for the critical temperature of the Higgs model with potential (5.1). Especially, for $m_{1p} = m_{2p}$ when the two fields decouple (5.6) becomes

$$T_c = \max_i \Big\{ \frac{8}{\alpha} \Phi_{ip} \Big\}, \tag{5.7}$$

in agreement with (4.14).

We conclude this section by indicating how one can extend our method to cases where the scalar field Φ has more than two components. It is not hard to see that the one-loop correction, which determines the renormalization flow, is still given by $V_{1-loop} = \frac{1}{2}(d^2 - 1)^{-1/2}\, tr\, V"$. The only calculational challenge is the computation of the λ-derivative of $W"(m+d-1, 0)$ at $\lambda = 0$. For its computation it helps to observe that $W"(m+d-1,0)$ is diagonal for an even classical potential and its diagonal elements are given by (B4). All one needs for their evaluation is Wick's theorem with respect to a gaussian measure like the one in (B5). One sees that nothing conceptually new is required in cases where the Higgs field has more components.

It would be very interesting to extend our method to coupled Yang-Mills-Higgs systems. Since the modified MF-approximation for pure lattice gauge systems at finite temperature is known [13], this extension, at least to abelian Higgs theories, presents itself.

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Appendix A

With the abbreviations

$$H_s = \sum_{sl\ NN} \phi_i\phi_j \qquad \text{and} \qquad H_t = \sum_{tl\ NN} \phi_i\phi_j \tag{A1}$$

where the first sum is over all spacelike (sl) separated NN and the second sum over all timelike (tl) separated NN, the actions S_ϵ in (2.12) read $S_\epsilon = -H_\epsilon + V_d = -H_s - \epsilon H_t + V_d$. One easily derives

$$\frac{d}{d\epsilon}\langle\phi_0\phi_i\rangle_\epsilon = \langle\phi_0\phi_i H_t\rangle_\epsilon - \langle\phi_0\phi_i\rangle_\epsilon\langle H_t\rangle_\epsilon. \tag{A2}$$

By doubling the fields, the right hand side can be written as

$$Z^{-2}\int\{\,\phi_0\phi_i(H_t[\phi] - H_t[\chi])\}\, e^{-S_\epsilon[\phi]-S_\epsilon[\chi]}\prod d\phi_j d\chi_j. \tag{A3}$$

To see that this expression is positive we change variables by an orthogonal transformation

$$\phi_i = \frac{1}{\sqrt{2}}(t_i + q_i) \qquad \text{and} \qquad \chi_i = \frac{1}{\sqrt{2}}(t_i - q_i),$$

in terms of which

$$\begin{aligned}
\phi_0\phi_i(H_t[\phi] - H_t[\chi]) &= \frac{1}{2}(t_0 + q_0)(t_i + q_i)\sum_{tl\ NN}(t_k q_l + q_k t_l)\\
H_\epsilon[\phi] + H_\epsilon[\chi] &= \sum_{sl\ NN}(t_i t_j + q_i q_j) \;+\; \epsilon\sum_{tl\ NN}(t_i t_j + q_i q_j)\\
V_d[\phi] + V_d[\chi] &= \sum_i\left\{(m+d)(t_i^2 + q_i^2) + \frac{1}{2}g(t_i^4 + 6t_i^2q_i^2 + q_i^4)\right\}.
\end{aligned} \tag{A4}$$

Next one expands $\phi_0\phi_i(H_t[\phi] - H_t[\chi])\exp\{H_\epsilon[\phi] + H_\epsilon[\chi]\}$ in a power serie and observes that all monoms in the t's and q's have positive coefficients. However, $V_d[\phi] + V_d[\chi]$ is an even function in both, the q and t variables, and hence only even powers in the expansion contribute after integration. Therefore, as a sum of manifestly positive terms, the right side in (A2) is positive.

Appendix B

In 3 dimensions the bare coupling constants and bare masses scale linearly with λ and thus the Schwinger function for a 2-component model on 2 time slices reads

$$W(m+d-1,j) = \log \int \{ e^{j_1(\phi_1+\phi_2)+j_2(\chi_1+\chi_2)} \\ e^{2(\phi_1\phi_2+\chi_1\chi_2)-d(\phi_1^2+\phi_2^2+\chi_1^2+\chi_2^2)-\lambda V[\phi,\chi]} \} \tag{B1}$$

where $j = (j_1, j_2)$ and the parameters in V are determined by the coefficients in the flow (4.11). For $\lambda = 0$ one finds $W = (4d-4)^{-1}(j_1^2 + j_2^2)$ and hence $\lambda^{-d}\Gamma$ in (4.6) has the leading term

$$\lambda^{-d}\Gamma(m+d-1, \lambda^{\frac{d}{2}-1}\Phi) \sim (d-1)\lambda^{-2}(\Phi_1^2 + \Phi_2^2), \tag{B2}$$

which cancels the first term on the right side of (4.6).
Next we evaluate the λ-derivative of

$$2W"(m+d-1,0) = \begin{pmatrix} \langle(\phi_1+\phi_2)^2\rangle & \langle(\phi_1+\phi_2)(\chi_1+\chi_2)\rangle \\ \langle(\phi_1+\phi_2)(\chi_1+\chi_2)\rangle & \langle(\chi_1+\chi_2)^2\rangle \end{pmatrix} \tag{B3}$$

at $\lambda = 0$. For example, the derivative of the 1-1 component is

$$\langle(\phi_1+\phi_2)^2\rangle_0\langle V[\phi,\chi]\rangle_0 - \langle(\phi_1+\phi_2)^2 V[\phi,\chi]\rangle_0, \tag{B4}$$

where $\langle ... \rangle_0$ is computed with the gaussian measure

$$d\mu(\phi) = \frac{1}{N}\, e^{-(\Psi, A\,\Psi)}, \qquad A = \begin{pmatrix} d & -1 & 0 & 0 \\ -1 & d & 0 & 0 \\ 0 & 0 & d & -1 \\ 0 & 0 & -1 & d \end{pmatrix}, \tag{B5}$$

where Ψ is the four component vector $(\phi_1, \phi_2, \chi_1, \chi_2)$. The expectation values in (B4) can easily be calculated with the help of Wicks theorem.

For the model (5.1) and its flow (5.4), we found after some arithmetic

$$\frac{d}{d\lambda} 2W"(m+d-1,0)|_{\lambda=0} = \frac{\alpha}{8\cdot 96}\begin{pmatrix} 3\frac{m_{1p}}{\Phi_{1p}^2} + \frac{\Delta m_p}{\Delta\Phi_p} & 0 \\ 0 & 3\frac{m_{2p}}{\Phi_{2p}^2} + \frac{\Delta m_p}{\Delta\Phi_p} \end{pmatrix}.$$

With $2W"(m+d-1, j=0, \lambda=0) = 1/2\, Id$ and $\Gamma"(0) = W"(0)^{-1}$, one finds that the λ-derivative at $\lambda = 0$ of $\Gamma"(0)$ is, up to a factor -16, equals to the λ-derivative at $\lambda = 0$ of $W"(0)$. Together with (B2) and (4.6) this establishes the asymptotic expansion (5.5) for the modified MF potential.

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Figure Captions

Fig.1: The mean field (MF) and Monte Carlo (MC) approximations to the constraint effective potential in 1 dimension resp. 4 dimensions for the classical potential V with mass $m = -7.5$ and coupling constant $g = 10$.

Fig.2: Ratios of the exact renormalization flow and the asymptotic flow (4.4) in 3 dimensions for various values of the scale parameter λ.

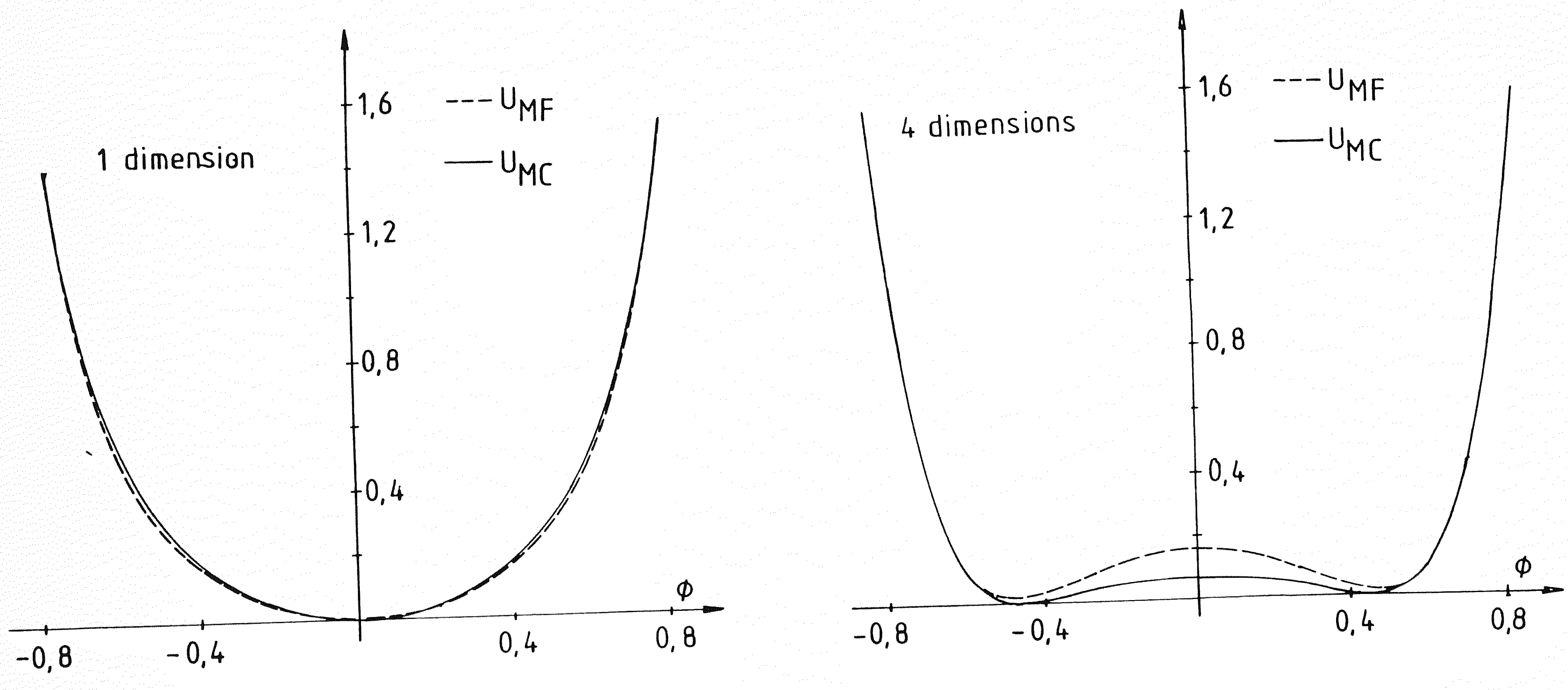
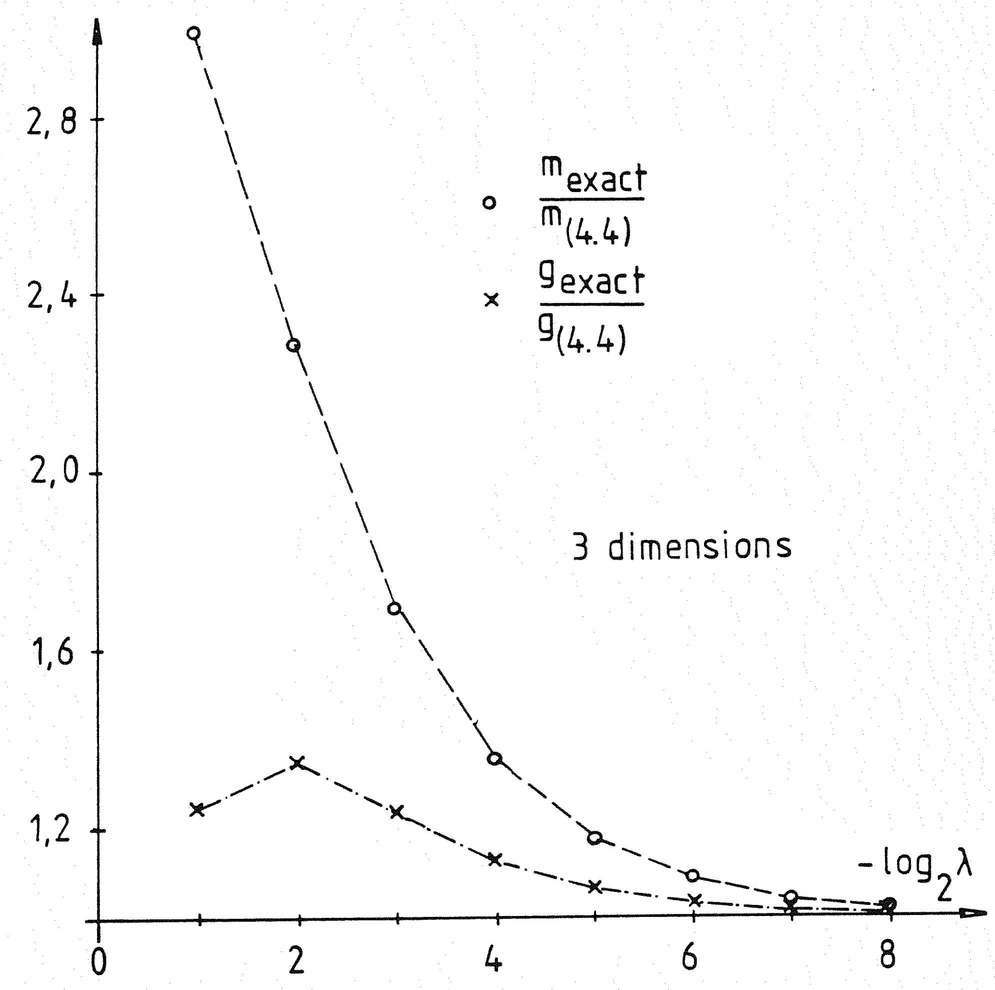

1 dimension
---UMF
UMC
1,6
1,2
0,8
0,4
φ
-0,8
-0,4
0,4
0,8
4 dimensions
---UMF
UMC
1,6
1,2
0,8
0,4
φ
-0,8
-0,4
0,4
0,8

FIGURE 1



2,8
2,4
2,0
1,6
1,2
0
2
4
6
8
$-\log_2\lambda$
$\frac{m_{exact}}{m_{(4.4)}}$
$\frac{g_{exact}}{g_{(4.4)}}$
3 dimensions

FIGURE 2