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MONOPOLE GEOGRAPHY *

P. A. Horváthy

Dublin Institute for Advanced Studies
10 Burlington Road, Dublin 4, Ireland

I. INTRODUCTION

The Yang - Mills - Higgs (YMH) action functional can be viewed as an "infinite - dimensional surface" over finite - energy field configurations whose critical points are the monopoles. For example, those charge-2 Bogomol'ny - Prasad - Sommerfield (BPS) monopoles [1,2] which represent two, widely separated monopoles form, in each topological sector, the absolute minima of the YMH functional. There exist however horizontal directions, in which such a configuration can evolve with no resistence, just like a ball could roll on such a surface (Fig. 1a). This is the geometric picture behind monopole scattering [3].

For a general non-Abelian monopole [1] the situation is slightly different: the YMH action has, generically, a saddle - like shape (Fig. 1.b)

If we put a ball into a "critical" (i.e. equilibrium) point on such a surface, it would be unstable and rather roll down. This property is reflected by the behaviour of the second derivative: if its matrix has a negative eigenvalue, there will be an unstable direction, and the number of independent instabilities coincides with the number of such negative eigenvalues.

Our analysis will show, that this is exactly what happens for monopoles : most of them are *unstable* [4,5], with the exception of the unique stable monopole [5,6,7] of each sector, for which the YMH

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functional looks rather like the bottom of a bowl (Fig. 1c).

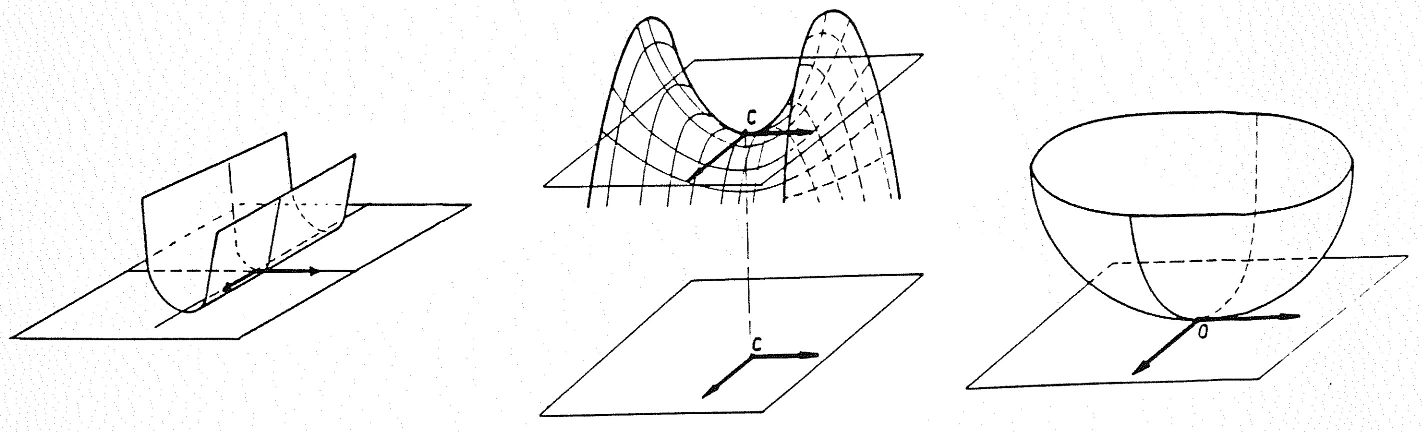

Fig 1. a   Fig. 1. b   Fig. 1. c

Another intuitive way of understanding monopole decay is by thinking of them as *elastic strings* wrapped around the residual group [5]. Such configuration are generally unstable, and will shrink rather to shorter ones and eventually to the shortest allowed by topology. In Section IV we translate this analogy into more rigourous terms. Combination of the two approaches provides us with information on *global aspects* of monopole decay (Section V).

Unstable monopoles can be thought of as excited states. They are nevertheless important, for they contribute to the Feynman integral when quantizing such a system. The instablity number ν should appear furthermore as a Maslov factor $e^{i\pi\nu/2}$ in the propagator.

II. FINITE - ENERGY CONFIGURATIONS [1,5]

Monopoles arise as static, everywhere - regular solutions to the YMH equations whose energy,

$$E = \int d^3x \, \{ \frac{1}{4} Tr(F_{ij}F^{ij}) + \frac{1}{2} Tr(D_i\Phi D^i\Phi) + V(\Phi) \} \qquad (1)$$

is finite. Although the results presented here are valid in full generality [7], we will restrict our attention to monopoles obtained in the spontaneous breaking of SU(3) by an adjoint Higgs field Φ wich has H =

U(2) (locally su(2)+u(1)) for residual group. Such monopoles, previously considered in the literature [8], contain already all the essential features of the general situation.

Finite energy implies non - trivial boundary conditions [1,5]. In particular, the Higgs field Φ maps $\mathbf{S}^2$, the "2-sphere at infinity", into the orbit G/H = SU(3)/U(2) $\simeq P_2(\mathbf{C})$. Such maps fall into topological sectors, labelled by homotopy classes in $\pi_2(P_2(\mathbf{C})) \simeq \pi_1(\mathrm{U}(2)) \simeq \mathbf{Z}$ i.e. by an integer "quantum number" m. Those sectors having different quantum numbers are separated by infinite energy barriers, and no transition is hence possible between them. In a fixed sector however, the different states are separated only by finite energy, and transitions are *a priori* not excluded. One of our aims is actually to give some information on possible routes between different configurations.

At large distances a monopole is is characterized by the so-called non - Abelian charge Q which is a constant vector in u(2) = su(2)+u(1), the Lie algebra of U(2). In fact, $\vec{A} = Q \cdot \vec{A}_D$ in a suitable (singular) gauge, where $\vec{A}_D$ is the vectorpotential of a Dirac monopole pf unit charge. To get a well -defined configuration, Q must be quantized, $exp\, 4\pi Q = 1$. Hence

$$h(t) = exp\, 4\pi Q t, \quad 0 \leq t \leq 1 \tag{2}$$

is a loop in U(2). Two monopoles belong furthermore to the same sector if and only if the corresponding loops (2) are homotopic in U(2) and have thus the same quantum number m.

The possibilities are shown on Fig. 2 : since any monopole charge Q is gauge - equivalent to a diagonal vector in u(2), we can represent Q by a point in the plane. The horizontal axis stands for diagonal su(2) matrices i.e. for multiples of σ_3, and the vertical axis is the central u(1). Quantization requires then that $2Q$ may only be one of the dotted points. Q_1 and Q_2 belong to the same homotopy sector if and only if they lie on the same horizontal line. Those vectors on the same vertical line (called a root plane) have the same (integer) eigenvalues in the adjoint representation. The pattern is remarkably periodic in the quantum number m modulo 2.

In each sector m there is an (up to conjugation unique) charge $\overset{\circ}{Q}$ which is the closest to the origin. It has eigenvalues 0 for m even or ± 1

for m odd. As we shall see, it represents the *unique stable* monopole in the sector, and all other configurations will be shown to be unstable.

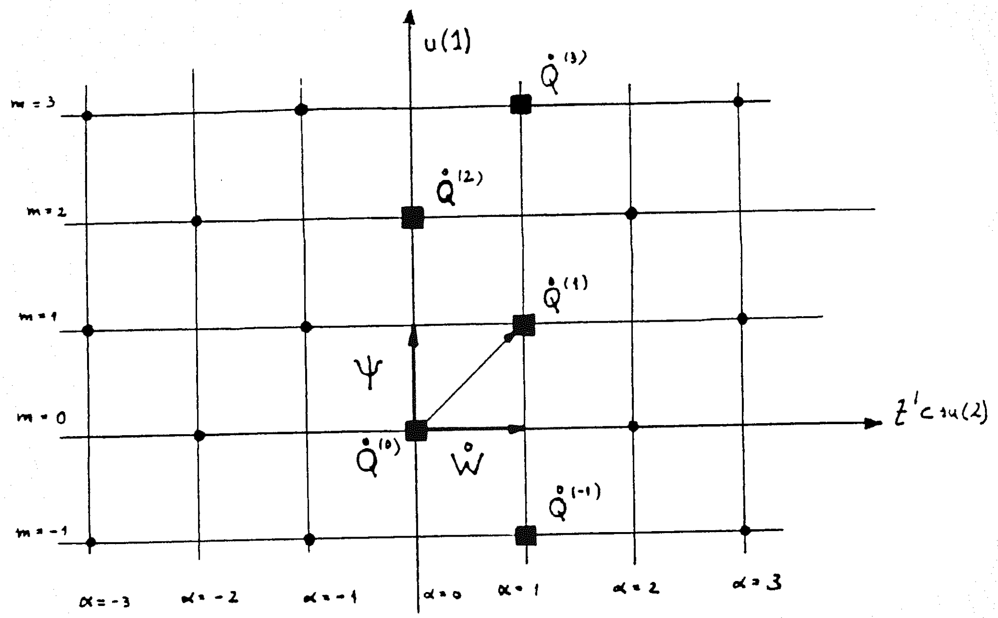

Fig. 2

III. NEGATIVE MODES [7]

In this section we count and explicitly construct the negative modes of a given monopole. We restrict our attention to "Brandt - Neri" type [4] variations i.e. such that $\delta\Phi = 0$ and $\vec{a} = \delta\vec{\mathbf{A}}$ is non-zero only outside the monopole core.

The second variation (called the Hessian) of the YMH functional (1) is expressed as

$$\delta^2 E = \int d^3x\, Tr\{(\vec{D}\times\vec{a})^2 + (\vec{B}\,\vec{a}\times\vec{a}) \;+\; ([\Phi,\vec{a}])^2\}. \qquad (3)$$

For non-zero Higgs potentials Φ tends to its asymptotic value $\Phi_0 = c\, diag\,(1,1,-2)$ exponentially; the third term drops out hence since $[\vec{a},\Phi_0] = 0$.

The radial contribution in $\vec{D}\times\vec{a}$ is $\partial_r\vec{a}$ (since $\vec{A}_0 = 0$) and yields a "mass" term

$$\delta^2 E_r = \int dr d\Omega\, Tr\,(\partial_r\vec{a})^2 = M^2\|\vec{a}\|^2. \qquad (4)$$

where $\|\vec{a}\|^2 = \int d^3x\, Tr\, \vec{a}^2$. The mass term is shown to be $\frac{1}{4} + \delta^2$ with δ^2 arbitrarily small for suitable $\vec{a}$.

Physical variations are furthermore orthogonal to gauge transformations $\vec{D}\chi$ i.e. satisfy

$$\vec{D} \cdot \vec{a} = 0. \tag{5}$$

$(\vec{D} \cdot \vec{a})^2 = 0$ can be added hence to the integrand in (3). This yields, after some transformations,

$$\delta^2 E = M^2 + \int dr d\Omega \left(\vec{J}^2 - \vec{b}(\vec{b} - 1)\vec{a}, \vec{a}\right) + \int d^3x\, Tr\left([\vec{B}, \vec{a}]\, \vec{a}\right), \tag{6}$$

where $\vec{b} = lim_{r \to \infty} r^2 \vec{B}(\vec{x})$ ($\vec{b} = Q$ in the Dirac gauge), and

$$\vec{J} = \vec{x} \times \vec{D} + \vec{x}(\vec{x} \cdot \vec{B}) \tag{7}$$

is the conserved *angular momentum.* Hence $\vec{J}^2 = j(j+1)$.

The variation $\vec{a}$ is conveniently decomposed as $\vec{a} = a_i \sigma_i + a_0 1$. Then $[\vec{b}, \sigma_3] = [\vec{b}, 1] = 0$ $[\vec{b}, \sigma_i] = \epsilon_{ij} q_j \sigma_j$ $(i, j = 1, 2)$ implies that the σ_3 direction is stable. For $\vec{a}$ parallel to $\sigma_1, q_i = q$ (say) we have in turn

$$\delta^2 E = \left[\left(M^2 + j(j+1) - q(q-1)\right) + q\right] \|\vec{a}\|^2. \tag{8}$$

The first bracket here is always non-negative so for having a negative variation we need $q < 0$. q is integer or half-integer; if $|q| \geq 1$, the angular momentum can take the values j = $|q|$- 1, $|q|$, $|q|$+ 1 ,.... For $j = |q| - 1$ $\;j(j+1) - q(q-1) = 0$ and $q \leq -1$ dominates $M^2 = \frac{1}{4} + \delta^2$, so we get a negative mode. For $j \geq |q|$ $\;j(j+1) - q(q-1) \geq 2\,|q|$and the modes are positive. Since the $j = |q| - 1$ states form SU(2) multiplets, we conclude that there are $2j + 1 = 2|q| - 1$ negative modes parallel to σ_1. Since σ_2 contributes with the same amount, we can conclude that our monopole has

$$\nu = 2(2|q| - 1) \tag{9}$$

negative modes. Notice that ν is always even. The number (9) is twice the number of root planes intersected by the straight line which joins $2Q$ to the origin (Fig. 2).

On the other hand, if

$$q = 0 \quad or \quad \pm 1/2, \tag{10}$$

there are no j = $|q|$- 1 states so that the Hessian is positive and the monopole is stable. This is the "Brandt - Neri" condition [4,5,6,7].

Using some knowledge from the structure theory of Lie algebras it is possible to prove [6,7] that each topological sector contains exactly one charge $\overset{\circ}{Q}$ whose only eigenvalues are 0 or $\pm 1/2$ i.e. a unique stable monopole. In our case this is

$$\overset{\circ}{Q} = mW_{(m)} + \frac{m}{2}1, \tag{11}$$

where (m) is m modulo 2, $W_0 = 0$ and $W_1 = \sigma_3/2$. Observe that W_1 is only "half of a charge", and the rule (11) tells us how one should add another "half of a charge" from the centre to it for getting the shortest charge in the sector.

For BPS monopoles the above argument breaks down: since Φ tends to its asymptotic value rather as $\Phi \sim \Phi_0 + Q/r$, an extra term q^2 comming from $([\Phi, \vec{a}])^2$ should be added to (8) which becomes thus positive: BPS monopoles are hence (Brandt - Neri) stable [4].

The *construction* of the negative modes relies on the observation that j = $|q|$- 1 states satisfy $(\vec{D} \times \vec{a})^2 + (\vec{D} \cdot \vec{a})^2 = 0$ i.e. are solutions to the coupled *first order* equations

$$\vec{D} \times \vec{a} = 0 \quad \vec{D} \cdot \vec{a} = 0. \tag{12}$$

They are easily found [7] to be

$$\vec{a} = \frac{1}{2} e^{i(k+1)\phi} (sin\theta/2)^k \, (cos\theta/2)^{2|q|-2-k} (d\theta + sin\theta d\phi) \tag{13}$$

where $0 \leq k \leq 2\,|q|-2$ is an integer, and θ, ϕ are polar coordinates.

IV. LOOPS IN H

Monopoles can be viewed as analogous to elastic strings wrapped around the residual group H [5]. Their decay is then reminding to the way how such configurations shrink to shorter ones.

Remarkably, this analogy can be made more rigourous [9,10]. Indeed, if we cover $\mathbf{S}^2$ with a 1 -parameter family of loops $\gamma_\phi, 0 \leq \phi \leq 2\pi$, to each YM connection $\vec{A}$ on $\mathbf{S}^2$ we can associate a loop $h^A(\phi)$ in H by parallel transport [1,9,10]:

$$h^A(\phi) = P\left(exp \oint_{\gamma_\phi} \vec{A}\right). \tag{14}$$

The loop $h^A(\phi)$ depends on the choice of the loops γ_ϕ only up to homotopy. A clever choice is to let γ_ϕ start from the south - pole S, follow upwards the meridian at angle $\phi/2$ until the north pole N and return then to S along the meridian at $-\phi/2$. Applied to $\vec{A} = \vec{A}_D$, a monopole, (14) yields exactly the loop (2) which is a *geodesic* in H.

Monopole theory can be mimicked for loops: let us defined in fact the *energy* of a loop $h(\phi)$ to be

$$L(h) = \frac{1}{4\pi}\int_0^1 Tr\ \left(h^{-1}\frac{dh}{d\phi}\right)^2 d\phi. \tag{15}$$

A loop - variation $\eta(\phi)$ is a loop in the Lie algebra such that $\eta(0) = \eta(2\pi) = 0$. A critical point $\delta L(h) = 0$ is exactly a geodesic. The second variation is calculated at once,

$$\delta^2 L(\eta,\eta) = -\frac{1}{4\pi}\int Tr\{(\frac{d^2\eta}{d\phi^2} + 2\pi[Q, \frac{d\eta}{d\phi}]) \cdot \eta\} dt, \tag{16}$$

whose negative modes are

$$\eta(\phi) = e^{-i|q|\phi}\left(e^{i(k+1)\phi/2} - e^{-i(k+1)\phi/2}\right)\sigma_i \tag{17}$$

where $0 \leq k \leq 2\ |q|$- 2.

We conclude that the map (14) carries monopoles (critical points of the YM functional) into geodesics (critical points of the loop - energy (15)); the energies of critical points are the same (namely $4\pi TrQ^2$), and the number of negative modes are also the same. With our clever choice of the γ_ϕ's we have even one more good property, namely that the image of a YM-negative mode under the differential of (14), namely

$$\eta^A(\phi) = -\oint g^{-1}(\theta,\phi)\vec{A}(\gamma_\phi(\theta))g(\theta,\phi)d\theta, \tag{18}$$

(where $g(\theta, \phi)$ is parallel transport along γ_ϕ from 0 to θ) is exactly a loop - negative mode (17), at least up to the numerical factor

$$C^k = \int_0^\pi (sin\theta/2)^k (cos\theta/2)^{2|q|-2-k}. \tag{19}$$

The map $\vec{A} \to h^A(\phi)$ is far from being (1-1); there is a full 1 - parameter family of YM configurations which go onto the same loop. One possible inverse is given however by [9,10]

$$A_\theta = 0, \quad A_\phi = \begin{cases} \left(\frac{1-cos\theta}{2}\right) h^{-1} \frac{dh}{d\phi} & in\ N \\ \left(\frac{1+cos\theta}{2}\right) \frac{dh}{d\phi} h^{-1} & in\ S \end{cases} \tag{20}$$

where N and S denote the upper and lower hemispheres.

V. GLOBAL ASPECTS

Since $E(\vec{A}+\vec{a}, \Phi) = E(\vec{A}, \Phi) + \frac{1}{2}\delta^2 E(\vec{a}, \vec{a}) + O(\vec{a}^3)$, variations of the form (13) reduce the energy and the monopole cannot be stable. But into what can it decay ? Again by analogy, consider a ball put on the top of a sphere (Fig. 3a) or of a torus (Fig. 3b). Such a position is unstable and the ball will roll down; it may choose any of the two independent directions. Following such a direction, it first arrives to another critical point; if it is again unstable, it continues to roll until it ends up in the final stable position. At the quantum level all such routes are simultaneously followed, and we get a kind of cascade.

Geometrically, following the negative modes for a while we get a small "cap" which, when glued to the surface of lower energy, forms a closed 2-surface. Morse theory [11] tells us that this surface is a generator of H_2 , the second homology group of configuration space (for background see, e.g., Refs. [12].

A similar picture is valid for monopoles: when following the negative modes we get a ν -dimensional energy - reducing surface with the (unstable) monopole at its top. This surface may have a quite complicated shape, but its critical points must be other (lower -energy) monopoles. The possible routes from our starting configuration to these intermediate one, and the subsequent routes from them indicate the likely way the monopole will decay.

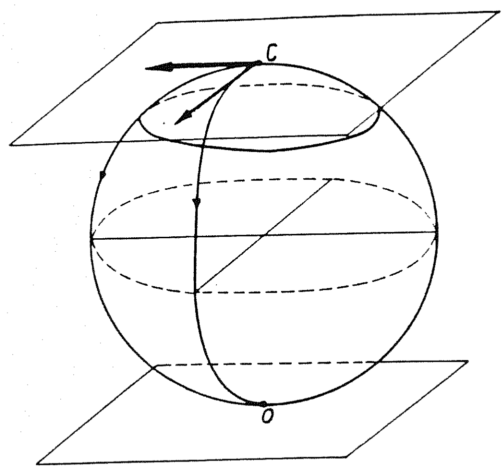

Fig. 3a

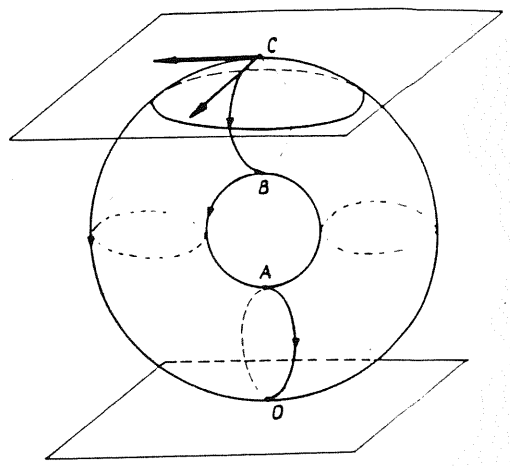

Fig. 3b

The first homology is zero : $H_1 \simeq \pi_1/[\pi_1, \pi_1] = 0$, which follows from $\pi_1(YM\ on\ \mathbf{S}^2) \simeq \pi_1(loops\ in\ \mathrm{H}) \simeq \pi_2\ (\mathrm{H}) = 0$ by Cartan 's theorem . So let us try H_2. $\pi_1 = 0$, and the Hurewicz isomorphism [12] tells then that $H_2 \simeq \pi_2$. We have furthermore the homotopy relation [13] (familiar in monopole theory [1])

$$\begin{aligned} &\pi_2(\{YM\,connections\}/\{gauge\,transformations\}) \\ &\simeq \pi_1(\{gauge\,transformations\}). \end{aligned} \tag{21}$$

But a gauge transformation is a map from $\mathbf{S}^2$ into H and hence

$$\begin{aligned} &\pi_1(\{gauge\,transformations\}) \simeq \pi_1\{Maps(\mathbf{S}^2 \to H)\} \\ &\simeq \pi_3(H) = \pi_3(U(2) \simeq \pi_3(SU(2)) \simeq \pi_3(\mathbf{S}^3) \simeq \mathbf{Z}. \end{aligned} \tag{22}$$

Another way of seeing this is by using the relation

$$\pi_2(YM\,on\,\mathbf{S}^2) \simeq \pi_2(loops\,in\,H) \simeq \pi_3(H), \tag{23}$$

which is a consequence of the map (14) being a homotopy equivalence.

In a more down - to - earth language, we want to construct two - spheres of YM configurations with the given monopole at their top.

Since the map (14) tells us that, topologically, "YM on $\mathbf{S}^2$ = loops in H", we need $\mathbf{S}^2$'s sitting in the loop - space of H = U(2). The geometric content of eqn. (23) is contained in Fig. 4 : a vertical plane which contains the vertical tangent to $\mathbf{S}^3$ at the "west -pole" cuts $\mathbf{S}^3$ in a circle, and when rotating the plane around the tangent, we get a family of circles parametrized by the "equator" $\mathbf{S}^2$ i.e. a two - sphere of loops.

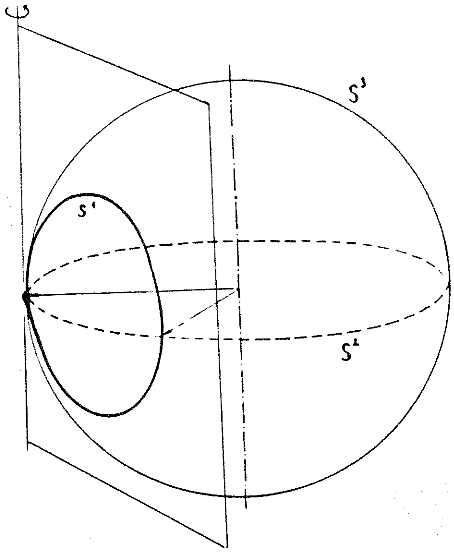

Fig. 4

Explicitly, the construction goes as follows: the orbit of σ_3 under U(2) is a two-sphere $\mathbf{S}$ in the Lie algebra u(2); observe that, for any vector $\xi \in \mathbf{S}$, $exp\, \pi \xi$, $= exp\, \pi(g \sigma_3 g^{-1}) = g(exp\, \pi \sigma_3) g^{-1} = -1$. Consequently

$$h_\xi(\phi) = \left[exp\, \phi(k+1)\xi/2\right] \left[exp\, \phi(2Q - (k+1)\sigma_3/2)\right] \qquad (24)$$

is a loop for each ξ . The speed $dh/d\phi$ of such a loop is constant, so its loop - energy (15) is calculated at once to yield

$$L(h) = \pi(k+1)(2|q| - k - 1)\left(\frac{1+z}{2}\right) + const, \qquad (25)$$

where z is the height - function on $\mathbf{S}^2$. So the energy behaves exactly as shown on Fig. 3a : as long as $0 \le k \le 2\,|q|$-2, it has a maximum at the top (i.e. for z = 1) which is just the "long" geodesic $exp\, 2Q\phi$, and a minimum at the bottom (i.e. for z = -1) which is a shorter geodesic, namely $exp\,[2Q-(k+1)\sigma_3/2]\phi$. We get hence $2|q|$-1 energy - reducing two -spheres, whose tangent vectors at the top,

$$\eta^k(\phi) = e^{2i|q|\phi}\big(e^{-i(2|q|-k-1)} - e^{-2|q|\phi}\big)\sigma_i \quad (i = 1, 2) \tag{26}$$

are negative modes of the loop - Hessian (16) (they are not, however, eigenmodes).

Having constructed $2|q|$-1 independent energy - reducing two - spheres, we get exactly the required number (9) negative modes. This shows that Fig. 3b is also valid for monopoles, when we replace "circles" by two - spheres. The product of our two -spheres is a ν dimensional surface which generates, by the Künneth formula [12] the ν -dimensional homology.

Finally, the inverse formula (20) translates these results into YM language: the images under (20) of our spheres are two - spheres of YM configurations which have the same energy as in (25).

The monopole studied in [8] for example has a charge Q conjugate to $\sigma_3/2$. It lies hence in the vacuum sector (it is actually the first charge on the right on the horizontal axis in Fig. 2). Q has eigenvalues ± 1 , and is so unstable with two negative modes.

$$\vec{A}_\xi = \frac{1-cos\theta}{4}\big(e^{-\phi\sigma_3/2}\xi e^{\phi\sigma_3/2} + \sigma_3\big)d\phi \tag{27}$$

is then an energy-reducing two-sphere of YM configurations on $\mathbf{S}^2$. For $\xi = \sigma_3$ we get

$$\vec{A} = (1 - cos\theta)\sigma_3\, d\phi/2 \tag{28}$$

i.e. the monopole we started with and for $\xi = -\sigma_3$ we get the vacuum.

VI. CONCLUSION

In the investigations presented here I have restricted myself to aysmptotic variations of the Yang - Mills field alone. There are, however, much more instabilities, namely those comming from the variations of the Higgs field. This implies Prasad - Sommerfield monopoles are also unstable, unless they satisfy the Bogomol'ny equation [14] ; also other models [15] contain Higgs - unstable solutions. Finally, the charge 5 sector of 't Hooft - Polyakov monopoles does not have a stable monopole [16]. This is again a consequence of the Higgs - instability.

Let me mention finally that Higgs - instablities seem to have also a derivation similar to that presented in Section V [17].

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REFERENCES

[1] Goddard P and Olive D (1978), Rep. Prog. Phys. **41**, 91;

[2] Forgács P , Horváth Z and Palla L (1981) Phys. Lett. **99B**, 232; Ward R (1981) Comm. Math. Phys. **79**, 317

[3] Manton N S (1986) Talk given at this Conference Fehér (1986) *ibid*

[4] Brandt R A and Neri F (1979) Nucl. Phys. **B161** 253

[5] Coleman S (1983) in The unity of fundamental interactions Ed. A. Zichichi, Plenum N.Y.

[6] Goddard P and Olive D (1981) Nucl. Phys. **B191** , 528

[7] Horváthy P A and Rawnsley J H (1986), in Proc. XVth Int. Conf. Diff. Geom. Meths. in Theor. Phys. Clausthal '86, Ed. Doebner, World Sci. (to be published); Horváthy P A and O'Raifeartaigh L (1986), Nucl. Phys. **B** (submitted)

[8] Corrigan E, Olive D I, Fairlie D B and Nuyts J (1976) Nucl. Phys. **B106** 475;
Burzlaff J (1981) Phys. Rev. **D23** , 1329

[9] Friedrich Th and Habermann L (1985) Comm. Math. Phys. **100** , 231

[10] Horváthy P A and Rawnsley J H (1986) in preparation

[11] Milnor J (1963) Morse Theory Annals of Math. Study **51**, Princeton Univ. Press (Princeton : N.J.);
Bott R (1982) Bull. Amer. Math. Soc.**7** (3) , 331

[12] Bott R and Tu L W (1982) Differential Forms in Algebraic Topology, Springer - Verlag N.Y., Heidelberg , Berlin

[13] Alvarez-Gaumé L and Ginsparg P (1985) Nucl. Phys. **B262** 439

[14] Taubes C (1982) Comm. Math. Phys. **86**, 257

[15] Manton N S (1983) Phys. Rev. **D28**, 2019;
Burzlaff J (1984) Nucl. Phys. **B233** , 262
Forgács P and Horváth Z (1984) Phys. Lett. **138** , 397

[16] Gardner C and Harvey J A (1984) Phys. Rev. Lett. **52**, 879

[17] Horváth Z and Palla L (1984) unpublished