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EXPLICIT CONSTRUCTION OF THE MASSIVE SUPERSYMMETRY MULTIPLETS ON SPACE-TIME

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ABSTRACT

A systematic method of constructing supersymmetry multiplets of second quantized fields is given for the massive case and for any spin, starting from the irreducible representations of the Poincaré Lie superalgebra. This allows a full understanding of the nature of the auxiliary fields.

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I. INTRODUCTION

The main object of this paper is to construct second quantized fields that transform according to representations of the proper orthochronous Lorentz group and which form supermultiplets, starting from the representation theory of the Lie superalgebra of the proper orthochronous Poincaré group. The analysis is given for the case of massive particles with $N=1$. Since this is a 'systematic' method it has advantages over the 'ad hoc' arguments that have been used previously.

In section II the theory of the irreducible representations of the Poincaré Lie superalgebra that was first given by Salam and Strathdee[1] (and which is described in most review articles on supersymmetry (eg. Fayet and Ferrara[2])) is developed further.

In section III we review the procedure for constructing second quantized fields transforming as some representation of the Lorentz group from some unitary representation of the Poincaré group, as detailed by Weinberg[3]. This approach has never previously been taken for supersymmetry theories.

In section IV we construct the left handed supermultiplets using the theory developed in sections III and IV. In section V we construct the right handed supermultiplets from the left handed set of section IV. In section VI we show how the phase factors of the fields can be altered so that the equations giving the action of the supersymmetry operators are symmetric under the interchange of L and R.

In section VII we examine methods of constructing combinations of these chiral supermultiplets in such a way that the fields obey equations other than the Klein-Gordon equation.

Appendix A gives our conventions for the Pauli and Dirac matrices and contains some identities used in the text. Appendix B is a short review of the super Poincaré algebra as used in this paper.

II. THE LITTLE ALGEBRA

The particle content of the supersymmetry multiplets is well known and was first given by Salam and Strathdee[1]. A supermultiplet consists of four particles of spins j, $j+½$, $j-½$, *and* j, except in the case when $j=0$, in which case the $j-½$ particle does not exist and we have just three particles in the supermultiplet. In this section we will establish the precise relationship between the rest states of these particles in the massive case. We do this using Clebsch-Gordan coefficients of $SU(2)$. The results given by Theorem II.1 have never previously been presented in the literature.

First we observe that the operator $P^2 = P_\sigma P^\sigma$ commutes with every generator of the superalgebra so that its eigenvalues serve as one label for the irreducible representations that we use. As usual we denote this eigenvalue by M^2c^2, where c is the velocity of light and M is interpreted as the rest mass of the particle. Here we consider only the possibility $M > 0$.

A second label comes from considering the 'superspin operator' $\boldsymbol{K}$ defined by

$$\boldsymbol{K} = (S_\sigma P_\rho - S_\rho P_\sigma)(S^\sigma P^\rho - S^\rho P^\sigma)$$

and which also commutes with every generator of the Poincaré superalgebra. Here

$$S_\sigma = \frac{1}{2}\epsilon_\sigma{}^{\beta\lambda\mu} M_{\beta\lambda} P_\mu + \frac{i\hbar}{4}(C^{-1}\gamma_5\gamma_\sigma)^{\alpha\gamma} Q_\alpha Q_\gamma .$$

The operator K has eigenvalues of the form $-M^4c^4\hbar^2 j(j+1)$, where $j = 0, \frac{1}{2}, 1, \ldots$, so that j provides a convenient second label for these irreducible representation. It has previously been noted by Sokatchev[4], who gave a version valid only in the Majorana representation of the Dirac matrices.

Now denote the particle states in the representation labeled by M and j by $|p,k,m\rangle$ for $k=j,\ j+\frac{1}{2},\ j-\frac{1}{2}$ and $m=k,k-1,\ldots,-k+1,-k$, and by $|p,j,m\rangle\rangle$ for the second state of spin j with $m = j,j-1,\ldots,-j+1,-j$, with $p=(p^1,p^2,p^3,p^4)$ in each case. Then consider a particular set of states within this representation, which we take to be the rest states with $p=(0,0,0,Mc)=\hat{p}$. This set of states is left invariant by the generators $\{M_{ij}, i,j=1,2,3; Q_\alpha\}$ so that we are looking for the representations of the 'little superalgebra' generated by $\{M_{ij}, Q_\alpha\}$ on these rest states. Let $|\hat{p},k,m\rangle$ for $k=j,j+\frac{1}{2},j-\frac{1}{2}$ be an eigenvector of J_3, P_α and J^2 such that

$$J^2|\hat{p},k,m\rangle = \hbar^2 k(k+1)|\hat{p},k,m\rangle \quad ,$$

$$J_3|\hat{p},k,m\rangle = \hbar m|\hat{p},k,m\rangle \quad ,$$

$$P_i|\hat{p},k,m\rangle = 0 \text{ for } i=1,2,3 \quad ,$$

and

$$P_4|\hat{p},k,m\rangle = Mc|\hat{p},k,m\rangle \quad ,$$

with similar expressions for $|\hat{p},j,m\rangle\rangle$. The main result of this section is then given by the following theorem.

THEOREM II.1

Suppose the rest states of the particles in a representation are denoted by $|\hat{p},j,m\rangle$, $|\hat{p},j+\frac{1}{2},m'\rangle$, $|\hat{p},j-\frac{1}{2},m''\rangle$ and $|\hat{p},j,m\rangle\rangle$, with spin values $j,j+\frac{1}{2},j-\frac{1}{2}$ and j respectively, and with $m=j,j-1,\ldots,-j+1,-j$; $m'=j+\frac{1}{2},j-\frac{1}{2},\ldots,\ -j+\frac{1}{2},-(j+\frac{1}{2})$ and $m''=j-\frac{1}{2},j-3/2,\ldots,-j+3/2,-(j-\frac{1}{2})$. Then (with an appropriate choice of the relative phases):

$$\left\{\frac{\hbar}{Mc}\right\}^{\frac{1}{2}} Q^\dagger_{Ln}|\hat{p},j,m\rangle = \left(\begin{matrix}\frac{1}{2} & j \\ n & m\end{matrix}\middle|\begin{matrix}j+\frac{1}{2} \\ m+n\end{matrix}\right)|\hat{p},j+\tfrac{1}{2},m+n\rangle + \left(\begin{matrix}\frac{1}{2} & j \\ n & m\end{matrix}\middle|\begin{matrix}j-\frac{1}{2} \\ m+n\end{matrix}\right)|\hat{p},j-\tfrac{1}{2},m+n\rangle \tag{1a}$$

$$\left\{\frac{\hbar}{Mc}\right\}^{1/2} Q^{\dagger}_{Ln}|\hat{p}, j+½, m+½\rangle = 2n \left(\begin{array}{cc|c} ½ & j & j+½ \\ -n & m+½+n & m+½ \end{array}\right) |\hat{p}, j, m+½+n\rangle\rangle \qquad (1b)$$

$$\left\{\frac{\hbar}{Mc}\right\}^{1/2} Q^{\dagger}_{Ln}|\hat{p}, j-½, m-½\rangle = -2n \left(\begin{array}{cc|c} ½ & j & j-½ \\ -n & m-½+n & m-½ \end{array}\right) |\hat{p}, j, m-½+n\rangle\rangle \qquad (1c)$$

$$\left\{\frac{\hbar}{Mc}\right\}^{1/2} Q^{\dagger}_{Ln}|\hat{p}, j, m\rangle\rangle = 0 \qquad (1d)$$

$$\left\{\frac{\hbar}{Mc}\right\}^{1/2} Q_{Ln}|\hat{p}, j, m\rangle = 0 \qquad (1e)$$

$$\left\{\frac{\hbar}{Mc}\right\}^{1/2} Q_{Ln}|\hat{p}, j+½, m+½\rangle = \left(\begin{array}{cc|c} ½ & j & j+½ \\ n & m+½-n & m+½ \end{array}\right) |\hat{p}, j, m+½-n\rangle \qquad (1f)$$

$$\left\{\frac{\hbar}{Mc}\right\}^{1/2} Q_{Ln}|\hat{p}, j-½, m+½\rangle = \left(\begin{array}{cc|c} ½ & j & j+½ \\ n & m+½-n & m+½ \end{array}\right) |\hat{p}, j, m+½-n\rangle \qquad (1g)$$

$$\left\{\frac{\hbar}{Mc}\right\}^{1/2} Q_{Ln}|\hat{p}, j, m\rangle\rangle = 2n\left\{\left(\begin{array}{cc|c} ½ & j & j+½ \\ -n & m & m-n \end{array}\right) |\hat{p}, j+½, m-n\rangle + \left(\begin{array}{cc|c} ½ & j & j-½ \\ -n & m & m-n \end{array}\right) |\hat{p}, j-½, m-n\rangle\right\} \qquad (1h)$$

Here $n = ½, -½$ and $\left(\begin{array}{cc|c} ½ & j & j+½ \\ n & m & m+n \end{array}\right)$ are Clebsch-Gordan coefficients of $SU(2)$ for which the phase conventions of Condon and Shortley have been employed (cf. Cornwell[5], chapter 12). If $j=0$ then the $j-½$ representation does not exist and the representation just has the rest state vectors $|\hat{p}, j, m\rangle$, $|\hat{p}, j+½, m'\rangle$ and $|\hat{p}, j, m\rangle\rangle$.

PROOF

We choose the particle states $|\hat{p}, j, m\rangle$ to be such that $Q_{Ln}|\hat{p}, j, m\rangle = 0$ for $n=½, -½$, and then consider the set of states $\{|\hat{p}, j, m\rangle, \left\{\frac{\hbar}{Mc}\right\}^{1/2} Q^{\dagger}_{Ln}|\hat{p}, j, m\rangle, \left\{\frac{\hbar}{Mc}\right\} Q^{\dagger}_{L½} Q^{\dagger}_{L-½}|\hat{p}, j, m\rangle\}$. The proof is then in several stages.

(i) $\left\{\frac{\hbar}{Mc}\right\}^{1/2} Q^{\dagger}_{Ln}|\hat{p}, j, m\rangle$:

Since $\left\{\frac{\hbar}{Mc}\right\}^{1/2} Q^{\dagger}_{L½}|\hat{p}, j, j\rangle$ is an eigenvector of J_3 with eigenvalue $\hbar(j+½)$ and of J^2 with eigenvalue $\hbar^2((j+½)+1)(j+½)$, we can choose the relative phase of $|\hat{p}, j+½, j+½\rangle$ to be such that

$$|\hat{p}, j+½, j+½\rangle = \left\{\frac{\hbar}{Mc}\right\}^{1/2} Q^{\dagger}_{L½}|\hat{p}, j, j\rangle$$

and let the phases of $|\hat{p}, j+½, m\rangle$, for $m = j+½-1, \ldots, -(j+½)$, be determined by repeated application of J_-. With the convention that

$$J_- |\hat{p}, j+½, m\rangle = \hbar\mu_m^{j+½} |\hat{p}, j+½, m-1\rangle$$

and $$\mu_m^{j+½} = \{((j+½)+m)((j+½)-m+1)\}^{½}$$

(cf. Cornwell[5] chapter 12), then

$$J_- Q^\dagger_{L½} |\hat{p}, j, m\rangle = \hbar Q^\dagger_{L-½} |\hat{p}, j, m\rangle + \hbar\mu_m^j \; Q^\dagger_{L½} |\hat{p}, j, m-1\rangle$$

and $$J_- Q^\dagger_{L-½} |\hat{p}, j, m\rangle = \hbar\mu_m^j Q^\dagger_{L-½} |\hat{p}, j, m-1\rangle.$$ Hence

$$|\hat{p}, j+½, j-½\rangle = \left\{\frac{\hbar}{Mc}\right\}^{½} \left\{ \frac{1}{\mu_{j+½}^{j+½}} Q^\dagger{}_{L-½} |\hat{p}, j, j\rangle + \frac{\mu_j^j}{\mu_{j+½}^{j+½}} \; Q^\dagger_{L½} |\hat{p}, j, j-1\rangle \right\}.$$

This can be written in terms of the Clebsch-Gordan coefficients of $SU(2)$ as:

$$|\hat{p}, j+½, j-½\rangle =$$

$$\left\{\begin{matrix} ½ & j \\ -½ & j \end{matrix}\middle| \begin{matrix} j+½ \\ j-½ \end{matrix}\right\} \left\{\frac{\hbar}{Mc}\right\}^{½} Q^\dagger_{L-½} |\hat{p}, j, j\rangle + \left\{\begin{matrix} ½ & j \\ ½ & j-1 \end{matrix}\middle| \begin{matrix} j+½ \\ j-½ \end{matrix}\right\} \left\{\frac{\hbar}{Mc}\right\}^{½} Q^\dagger_{L½} |\hat{p}, j, j-1\rangle. \qquad (2)$$

Repeating this analysis we obtain

$$|\hat{p}, j+½, m\rangle =$$

$$\left(\begin{matrix} ½ & j \\ -½ & m+½ \end{matrix}\middle| \begin{matrix} j+½ \\ m \end{matrix}\right) \left\{\frac{\hbar}{Mc}\right\}^{½} Q^\dagger_{L-½} |\hat{p}, j, m+½\rangle + \left(\begin{matrix} ½ & j \\ ½ & m-½ \end{matrix}\middle| \begin{matrix} j+½ \\ m \end{matrix}\right) \left\{\frac{\hbar}{Mc}\right\}^{½} Q^\dagger_{L½} |\hat{p}, j, m-½\rangle. \qquad (3)$$

Now $Q^\dagger_{Ln} |\hat{p}, j, m\rangle$ is not an eigenvector of J^2 with eigenvalue $\hbar^2((j-½)+1)(j-½)$ for any n or m, but we can write

$$|\hat{p}, j-½, j-½\rangle = \alpha \left\{\frac{\hbar}{Mc}\right\}^{½} Q^\dagger_{L½} |\hat{p}, j, j-1\rangle + \beta \left\{\frac{\hbar}{Mc}\right\}^{½} Q^\dagger_{L-½} |\hat{p}, j, j\rangle \qquad (4)$$

with α, β chosen such that

$$J_+ |\hat{p}, j-½, j-½\rangle = 0.$$

Then by an analysis similar to the above we find that

$$|\hat{p}, j-½, m\rangle =$$

$$\left(\begin{matrix} ½ & j \\ -½ & m+½ \end{matrix}\middle| \begin{matrix} j-½ \\ m \end{matrix}\right) \left\{\frac{\hbar}{Mc}\right\}^{½} Q^\dagger_{L-½} |\hat{p}, j, m+½\rangle + \left(\begin{matrix} ½ & j \\ ½ & m-½ \end{matrix}\middle| \begin{matrix} j-½ \\ m \end{matrix}\right) \left\{\frac{\hbar}{Mc}\right\}^{½} Q^\dagger{}_{L½} |\hat{p}, j, m-½\rangle. \qquad (5)$$

Let $C_{nm,r}$ be the $2(2j+1)\times 2(2j+1)$ matrix of Clebsch-Gordan coefficients $SU(2)$ defined by

$$C_{nm,r} \begin{bmatrix} D^{j+½}(a) & 0 \\ 0 & D^{j-½}(a) \end{bmatrix}_{rr'} (C^{-1})_{r',n'm'} = D^{½}(a)_{nn'} \times D^j(a)_{mm'} \qquad (6)$$

with $a \epsilon SU(2)$ and $D^k(a)$ the irreducible representation of $SU(2)$ in $(2k+1)\times(2k+1)$ matrix form. Then equations (2), (3), (4) and (5) can be combined to give

$$(|\hat{p},j+\tfrac{1}{2},a'\rangle,\ |\hat{p},j-\tfrac{1}{2},a'\rangle)_r = C^{-1}_{r,nm} \left\{\frac{\hbar}{Mc}\right\}^{\frac{1}{2}} Q^{\dagger}_{Ln}\ |\hat{p},j,m\rangle.$$

Inverting this we obtain

$$\left\{\frac{\hbar}{Mc}\right\}^{\frac{1}{2}} Q^{\dagger}_{Ln}\ |\hat{p},j,m\rangle = C_{nm,r}(|\hat{p},j+\tfrac{1}{2},a\rangle,\ |\hat{p},j-\tfrac{1}{2},a'\rangle)_r \quad ,$$

which is the first of our required results.

(ii) $Q^{\dagger}_{L\alpha}Q^{\dagger}_{L\beta}|\hat{p},j,m\rangle$:

We define

$$|\hat{p},j,m\rangle\rangle = Q^{\dagger}_{L\frac{1}{2}}Q^{\dagger}_{L-\frac{1}{2}} \left\{\frac{\hbar}{Mc}\right\}\ |\hat{p},j,m\rangle.$$

and noting that the Clebsch-Gordan coefficients must satisfy

$$\begin{pmatrix} \frac{1}{2} & j & j+\frac{1}{2} \\ \frac{1}{2} & m & m+\frac{1}{2} \end{pmatrix}^2 + \begin{pmatrix} \frac{1}{2} & j & j-\frac{1}{2} \\ \frac{1}{2} & m & m+\frac{1}{2} \end{pmatrix}^2 = 1 \quad , \tag{8}$$

we obtain the action of $Q^{\dagger}_{Ln}$ on $|\hat{p},j+\frac{1}{2},m\rangle$ and $|\hat{p},j-\frac{1}{2},m\rangle$.

Lastly we note that $Q^{\dagger}_{Ln}|\hat{p},j,m\rangle = 0$.

(iii) The action of the operators Q_{Ln} :

We note that

$$[Q_{Ln}, Q^{\dagger}_{Ln'}] = \frac{Mc}{\hbar}\,\delta_{nn'}$$

when acting on the rest states, so that

$$Q_{Ln}Q^{\dagger}_{Ln'} = \frac{Mc}{\hbar}\,\delta_{nn'} - Q_{Ln'}Q^{\dagger}_{Ln}.$$

The proof proceeds as before, making use of equation (8).

This completes the proof of the theorem. It is convenient in the next section to have, in addition, some of these formulae expressed in terms of the operators Q_{Rn} as follows:

$$\left\{\frac{\hbar}{Mc}\right\}^{\frac{1}{2}} Q_{Rn}|\hat{p},j+\tfrac{1}{2},m+n\rangle = \begin{pmatrix} \frac{1}{2} & j & j+\frac{1}{2} \\ n & m & m+n \end{pmatrix} |\hat{p},j,m\rangle\rangle, \tag{9a}$$

$$\left\{\frac{\hbar}{Mc}\right\}^{1/2} Q_{Rn} |\hat{p}, j-1/2, m+n\,\rangle = \left(\begin{matrix} 1/2 & j \\ n & m \end{matrix}\middle| \begin{matrix} j-1/2 \\ m+n \end{matrix}\right) |\hat{p}, j, m\,\rangle\!\rangle, \tag{9b}$$

and
$$\left\{\frac{\hbar}{Mc}\right\}^{1/2} Q_{Rn} |\hat{p}, j, m\,\rangle\!\rangle = \left(\begin{matrix} 1/2 & j \\ n & m \end{matrix}\middle| \begin{matrix} j+1/2 \\ m+n \end{matrix}\right) |\hat{p}, j+1/2, m+n\,\rangle\!\rangle, + \left(\begin{matrix} 1/2 & j \\ n & m \end{matrix}\middle| \begin{matrix} j-1/2 \\ m+n \end{matrix}\right) . |\hat{p}, j-1/2, m+n\,\rangle. \tag{9c}$$

We note that each of the vectors $|\hat{p},j,m\rangle$, $|\hat{p},j+1/2,m'\rangle$, $|\hat{p},j+1/2,m''\rangle$ and $|\hat{p},j,m\rangle\!\rangle$ is needed in the representation derived above and no other vector is needed.

III. THE CONSTRUCTION OF SECOND QUANTIZED FIELDS

In this section we give a review of the construction of operator-valued fields that are Lorentz invariant starting from the irreducible representations of the Poincaré group. The method we use is that developed by Weinberg[3].

The unitary irreducible representation of the proper orthochronous Poincaré group corresponding to a particle of mass M and spin j with $\hat{p}=(0,0,0,Mc)$ is given by:

$$\mathcal{U}^{\hat{p},j}([\Lambda|t])\phi_{p,m} = exp(\, (\tfrac{i}{\hbar}\Lambda p)_{\sigma} t^{\sigma}) D^{j}([B(\Lambda p,\hat{p})^{-1}B(p,\hat{p})|0])_{m'm}\,\phi_{\Lambda p,m'}, \tag{10}$$

for $m=-j,-j+1,\ldots,j-1,j$. Here $\phi_{p,m}$ are the vectors of the carrier space of the representation, $\mathcal{U}^{\hat{p},j}([\Lambda|t])$ are the operators, for a given $\hat{p},j$, corresponding to an element $[\Lambda|t]$ of the covering group of the proper orthochronous Poincaré group, D^{j} is the $(2j+1)\times(2j+1)$ dimensional representation of the rotation group and $B(p,\hat{p})$ is the Lorentz boost from a 'rest state' labeled by $\hat{p}$ to a general state labeled by p, the the combination $B(\Lambda p,\hat{p})^{-1}\Lambda B(p,\hat{p})$ being then a pure rotation known as the

'Wigner Rotation'. (For more details of this formula and its derivation see Cornwell[5]).

Now we identify the one particle state $|p,j,m\rangle$, as used in the previous section with $\left(\frac{p_4}{Mc}\right)^{\frac{1}{2}} \phi_{p,m}$ and by comparison with equation (10) we define the unitary operator $U([\Lambda|t])$ by:

$$U([\Lambda|t])|p,j,m\rangle = \left\{\frac{(\Lambda p)_4}{p_4}\right\}^{\frac{1}{2}} \exp(\tfrac{i}{\hbar}(\Lambda p)_\sigma t^\sigma) D^j([B(\Lambda p,\hat{p})^{-1}\Lambda B(p,\hat{p})|0])_{m'm}|\Lambda p,j,m'\rangle. \tag{11}$$

Here and in the subsequent equations the repeated index m' is summed over all the values $j, j-1, \ldots, -j$.

Next we introduce the single particle creation operators $a^\dagger_{p,j,m}$ and the and the vacuum state $|0\rangle$ by:

$$|p,j,m\rangle = a^\dagger_{p,j,m}|0\rangle$$

and suppose that

$$U([\Lambda|t])|0\rangle = |0\rangle \quad \text{for all } [\Lambda|t].$$

Equation (11) can now be written

$$U([\Lambda|t])a^\dagger_{p,j,m}U([\Lambda|t])^{-1} = \left\{\frac{(\Lambda p)_4}{p_4}\right\}^{\frac{1}{2}} \exp(\tfrac{i}{\hbar}(\Lambda p)_\sigma t^\sigma) D^j([B(\Lambda p,\hat{p})^{-1}\Lambda B(p,\hat{p})|0])_{m'm}\, a^\dagger_{\Lambda p,j,m'}. \tag{12}$$

In particular we note that

$$a^\dagger_{p,j,m} = \left\{\frac{Mc}{p_4}\right\}^{\frac{1}{2}} U([B(p,\hat{p})|0])a^\dagger_{\hat{p},j,m}U([B(p,\hat{p})|0])^{-1}. \tag{13}$$

Now we define the corresponding annihilation operators $a_{p,j,m}$ by

$$a_{p,j,m} = (a^\dagger_{p,j,m})^\dagger$$

and take the adjoint of equation (12) to obtain:

$$U([\Lambda|t])a_{p,j,m}U([\Lambda|t])^{-1} = \left\{\frac{(\Lambda p)_4}{p_4}\right\}^{\frac{1}{2}} \exp(-\tfrac{i}{\hbar}(\Lambda p)_\sigma t^\sigma) D^j([B(\Lambda p,\hat{p})^{-1}\Lambda B(p,\hat{p})|0])^*_{m'm}\, a_{\Lambda p,j,m'}.$$

Since the matrices D^j are unitary we can rewrite this expression as:

$$U([\Lambda|t])a_{p,j,m}U([\Lambda|t])^{-1} = \tag{14}$$
$$\left\{\frac{(\Lambda p)_4}{p_4}\right\}^{1/2} \exp(-\tfrac{i}{\hbar}(\Lambda p)_\sigma t^\sigma)D^j([B(\Lambda p,\hat{p})^{-1}\Lambda B(p,\hat{p})|0])_{mm'}\, a_{\Lambda p,j,m'} .$$

It is convenient to rewrite equation (12) so that the indices are in the same order as those of equation (14). To do this we recall that since the representations D^j of $SU(2)$ are real or pseudo real (cf. Cornwell[5] Chapter 5) there exists a $(2j+1)\times(2j+1)$ matrix Z_j such that:

$$D^j([R|0])^* = Z_j^{-1}D^j([R|0])Z_j \tag{15}$$

Here $[R|0]$ denotes an element of $SU(2)$, the covering group of $SO(3,\mathbb{R})$, that maps onto the pure rotation R in the $2:1$ homeomorphic mapping of $SU(2)$ onto $SO(3,\mathbb{R})$.

We note that the matrix Z_j can be chosen to satisfy

$$Z_j^{-1} = Z_j^\dagger \tag{16}$$

and

$$Z_j^* Z_j = (-1)^{2j}. \tag{17}$$

For $j=\frac{1}{2}$ we put $Z_{1/2} = \sigma_2$. Equation (12) can now be written

$$U([\Lambda|t])a^\dagger_{p,j,m}\, U([\Lambda|t])^{-1} = \tag{18}$$
$$\left\{\frac{(\Lambda p)_4}{p_4}\right\}^{1/2} \exp(\tfrac{i}{\hbar}(\Lambda p)_\sigma t^\sigma)(Z_j^{-1}D^j([B(p,p)^{-1}\Lambda^{-1}B(\Lambda p,p)|0])Z_j)_{mm'}a^\dagger_{\Lambda p,j,m'} .$$

Next we assume that if the particle has a corresponding antiparticle its creation and annihilation operators $b^\dagger_{p,j,m}$ and $b_{p,j,m}$ have the same transformation properties as $a^\dagger_{p,j,m}$ and $a_{p,j,m}$ respectively.

We can now construct Lorentz invariant fields defined on space-time using these equations but it is advantageous to insert an intermediate step into the construction. To this end we let $\Gamma^{(j)}([\Lambda|0])$ be a $(2j+1)\times(2j+1)$ dimensional representation of the orthochronous Lorentz group that coincides with D^j when Λ is a rotation. Thus $\Gamma^{(j)}$ can be either $\Gamma^{0,j}$, the 'right handed' representation, or $\Gamma^{j,0}$, the 'left handed' representation. We note that $\Gamma^{(j)}$ is not a unitary representation. Then we define the ancilliary operators $\alpha_{p,j,m}$ and $\beta_{p,j,m}$ by

$$\alpha_{p,j,m} = \left\{\frac{2p_4}{Mc}\right\}^{1/2} \Gamma^{(j)}([B(p,\hat{p})|0])_{mm'} a_{p,j,m'} \tag{19}$$

and
$$\beta_{p,j,m} = \left\{\frac{2p_4}{Mc}\right\}^{1/2} \Gamma^{(j)}([B(p,\hat{p})|0]Z_j)_{mm'} b^{\dagger}_{p,j,m'} . \tag{20}$$

The transformation properties of these operators are

$$U([\Lambda|t])\alpha_{p,j,m} U([\Lambda|t])^{-1} = \exp\{-\tfrac{i}{\hbar}(\Lambda p)_\sigma t^\sigma\}\Gamma^{(j)}([\Lambda^{-1}|t])_{mm'}\alpha_{\Lambda p,j,m'} \tag{21}$$

and

$$U([\Lambda|t])\beta_{p,j,m} U([\Lambda|t])^{-1} = \exp\{\tfrac{i}{\hbar}(\Lambda p)_\sigma t^\sigma\}\Gamma^{(j)}([\Lambda^{-1}|t])_{mm'}\beta_{\Lambda p,j,m'} . \tag{22}$$

We note that if the particle under consideration is its own antiparticle then $a^{\dagger}_{p,j,m} = b_{p,j,m}$ but $\alpha^{\dagger}_{p,j,m} \neq \beta_{p,j,m}$ unless $j=0$.

Finally to obtain the field for a particle of spin j transforming as the $(2j+1)$ dimensional representation $\Gamma^{(j)}$ of the orthochronous Lorentz group we define

$$\chi_{j,m}(x) = \left\{\frac{1}{2\pi\hbar}\right\}^3 \int dp^3 \frac{Mc}{2p_4} \{ \rho\alpha_{p,j,m} e^{-\frac{i}{\hbar} p.x} + \rho'\beta_{p,j,m} e^{\frac{i}{\hbar} p.x} \} \tag{23}$$

with ρ, ρ' complex numbers such that |ρ|=|ρ'|=1. This field then has the transformation property

$$U([\Lambda|t])\chi_{j,m}(x)U([\Lambda|t])^{-1} = \Gamma^{(j)}([\Lambda^{-1}|0])_{mm'} \chi_{j,m'}(\Lambda x+t) . \tag{24}$$

IV. THE LEFT HANDED SUPERMULTIPLETS

We assume that $Q_\alpha|0\rangle=0$ so that we can express the action of Q_α on our rest state creation operators as

$$\left(\frac{\hbar}{Mc}\right)^{1/2}[\,Q_{\alpha'},\ a^{\dagger}_{\hat{p},k,m}] = M(Q_{\alpha'})_{m'k',mk}\ a^{\dagger}_{\hat{p},k',m'}\ . \tag{25}$$

where [,] is a commutator if k is an integer but an anti-commutator if k is a half integer, that is, the parity of the creation and annihilation operators is defined by

$$|a_{p,k,m}| = |a^{\dagger}_{p,k,m}| = (-1)^{2k}.$$

Here $M(Q_\alpha)_{m'k',mk}$ is a matrix whose coefficients can be determined from Theorem II.1. If we order the operators so that

$$a^{\dagger}_{p,k,m} = (a^{\dagger}_{p,j,m}\,,a^{\dagger}_{p,j+½,m}\,,a^{\dagger}_{p,j-½,m}\,,\underset{\sim}{a}^{\dagger}_{p,j,m})\quad,\ k = j,j+½,j-½,j$$

with $\quad a^{\dagger}_{p,k,m}|0\rangle = |p,k,m\rangle$ for $k = j,j+½,j-½$

and $\quad \underset{\sim}{a}^{\dagger}_{p,j,m}|0\rangle = |p.j,m\,\rangle\!\rangle$,

then the matrices $M(Q_\alpha)$ can be written

$$M(Q^{\dagger}_{Ln})_{m'k',mk} =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ \left(\begin{array}{cc|c} ½ & j & j+½ \\ n & m & m+n \end{array}\right) & 0 & 0 & 0 \\ \left(\begin{array}{cc|c} ½ & j & j-½ \\ n & m & m+n \end{array}\right) & 0 & 0 & 0 \\ 0 & 2n\left(\begin{array}{cc|c} ½ & j & j+½ \\ -n & m+½+n & m+½ \end{array}\right) & 2n\left(\begin{array}{cc|c} ½ & j & j-½ \\ -n & m+½+n & m+½ \end{array}\right) & 0 \end{bmatrix}_{m'k',mk} \tag{26}$$

and

$$M(Q_{Ln})_{m'k',mk} =$$

$$\begin{bmatrix} 0 & \left(\begin{array}{cc|c} ½ & j & j+½ \\ n & m+½-n & m+½ \end{array}\right) & \left(\begin{array}{cc|c} ½ & j & j-½ \\ n & m+½-n & m+½ \end{array}\right) & 0 \\ 2n\left(\begin{array}{cc|c} ½ & j & j+½ \\ n & m & m+n \end{array}\right) & 0 & 0 & 0 \\ 2n\left(\begin{array}{cc|c} ½ & j & j-½ \\ n & n & m+n \end{array}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{m'k',mk} \tag{27}$$

Now combining equation (25) with equation (13) we obtain

$$\left\{\frac{\hbar}{Mc}\right\}^{½}[Q_\alpha, a^\dagger_{p,k,m}] =$$

$$\left\{\frac{\hbar}{p_4}\right\}^{½} U(\lbrack B(p,\hat{p})|0\rbrack)\lbrack U(B\lbrack (p,\hat{p})|0\rbrack)^{-1} Q_\alpha U(\lbrack B(p,\hat{p})|0\rbrack), a^\dagger_{p,k,m}\rbrack U(\lbrack B(p,\hat{p})|0\rbrack)^{-1}. \tag{28}$$

To proceed we need the action of the operator correseponding to a boost on the supersymmetry generators Q_α. Since we also need to work in two component form we make this conversion at the same time. Equations (B3a) and (B3b) of the Appendix B imply that

$$U(\lbrack \Lambda| t\rbrack)^{-1} Q_{Ln} U(\lbrack \Lambda| t\rbrack) = \Gamma^{0,½}(\lbrack \Lambda|0\rbrack)_{nn'} Q_{Ln'} \tag{29a}$$

and $$U(\lbrack \Lambda| t\rbrack)^{-1} Q_{Rn} U(\lbrack \Lambda / t\rbrack) = \Gamma^{½,0}(\lbrack \Lambda|0\rbrack)_{nn'} Q_{Rn'} . \tag{29b}$$

Here and in the subsequent equations the repeated index n' is summed over +½ and -½.

Then combining equations (28) and (29) we find that

$$\left\{\frac{\hbar}{Mc}\right\}^{½}[Q_{Ln}, a^\dagger_{p,k,m}] = \Gamma^{0,½}(\lbrack B(p,\hat{p})|0\rbrack)_{nn'} M(Q_{Ln'})_{m'k',mk}\, a^\dagger_{p,k,m'} \tag{30a}$$

and $$\left\{\frac{\hbar}{Mc}\right\}^{½}[Q_{Rn}, a^\dagger_{p,k,m}] = \Gamma^{½,0}(\lbrack B(p,\hat{p})|0\rbrack)_{nn'} M(Q_{Rn'})_{m'k',mk}\, a^\dagger_{p,k,m'} \tag{30b}$$

Similarly if we put

$$\left\{\frac{\hbar}{Mc}\right\}^{½}[Q_\alpha, a_{\hat{p},k,m}] = N(Q_\alpha)_{m'k',mk}\, a_{\hat{p},k',m'} . \tag{31}$$

we find that

$$\left\{\frac{\hbar}{Mc}\right\}^{½}[Q_{Ln}, a_{p,k,m}] = \Gamma^{0,½}(\lbrack B(p,\hat{p})|0\rbrack)_{nn'} N(Q_{Ln'})_{m'k',mk}\, a_{p,k,m'} \tag{32a}$$

and $$\left\{\frac{\hbar}{Mc}\right\}^{½}[Q_{Rn}, a_{p,k,m}] = \Gamma^{½,0}(\lbrack B(p,\hat{p})|0\rbrack)_{nn'} N(Q_{Rn'})_{m'k',mk}\, a_{p,k,m'} \tag{32b}$$

The transformation properties of the antiparticle creation and annihilation operators $b_{p,k,m}$ and $b^\dagger_{p,k,m}$ are obtained simply by replacing a with b in equations (30) and (32).

To determine the coefficients of the matrices $N(Q_{Ln})$ and $N(Q_{Rn})$ we take the adjoint of equation (25) as follows.

Since $\left\{\frac{\hbar}{Mc}\right\}^{\frac{1}{2}} [\, Q_\alpha \,, a^\dagger_{p,k,m}] = M(Q_\alpha)_{m'k',mk}\; a^\dagger_{p,k',m'}$

then $\left\{\frac{\hbar}{Mc}\right\}^{\frac{1}{2}} [\, a_{p,k,m} \,, Q^\dagger_\alpha] = a_{p,k,m}\;\; M(Q_\alpha)_{m'k',mk}$

and so $(-1)^{2j} \left\{\frac{\hbar}{Mc}\right\}^{\frac{1}{2}} [\, Q^\dagger_\alpha \,, a_{p,k,m}] = M(Q_\alpha)_{m'k',mk}\; a_{p,k,m}$.

Thus

$$N(Q_{Ln})_{m'k',mk} = (-1)^{2j} (i\sigma_2)_{nn'} M(Q_{Rn'})_{m'k',mk} \tag{33a}$$

and

$$N(Q_{Rn})_{m'k',mk} = (-1)^{2j} (-i\sigma_2)_{nn'} M(Q_{Ln'})_{m'k',mk} \; . \tag{33b}$$

Next we define the ancilliary operators, following equations (19) and (20). We choose these to be all left handed. Thus:

$$\alpha^L_{p,k,\bar{\bar{m}}} = \left\{\frac{2p_4}{Mc}\right\}^{\frac{1}{2}} \Gamma^{0,k}([\, B(p,\hat{p})|0]\,)_{mm'}\; a_{p,k,m'} \tag{34a}$$

for $k=j.\, j+\frac{1}{2}, j-\frac{1}{2}$ and

$$\text{and} \qquad \alpha^L_{\sim p,j,\bar{\bar{m}}} = \left\{\frac{2p_4}{Mc}\right\}^{\frac{1}{2}} \Gamma^{0,j}([\, B(p,\hat{p})|0]\,)_{mm'}\; a_{\sim p,j,m'} \; . \tag{34b}$$

Similarly

$$\beta^L_{p,k,\bar{\bar{m}}} = \left\{\frac{2p_4}{Mc}\right\}^{\frac{1}{2}} \Gamma^{0,k}([\, B(p,p)|0]\, Z_k)_{mm'}\; b^\dagger_{p,k,m'} \tag{34c}$$

for $k=j.\, j+\frac{1}{2}, j-\frac{1}{2}$ and

$$\beta^L_{\sim p,j,\bar{\bar{m}}} = \left\{\frac{2p_4}{Mc}\right\}^{\frac{1}{2}} \Gamma^{0,j}([\, B(p,p)|0]\, Z_j)_{mm'}\; b^\dagger_{\sim p,j,m'} \; . \tag{34d}$$

To proceed we rewrite equations (30) and (32) in terms of the ancilliary operators and reduce the resulting equations. The analysis is very similar in each case and relies on the following Lemma, and the identities given in Appendix A.

LEMMA IV.1

The Clebsch-Gordan matrix $C_{nm,r}$ defined by

$$C_{nm,r}\begin{bmatrix} D^{j+\frac{1}{2}}(R) & 0 \\ 0 & D^{j-\frac{1}{2}}(R) \end{bmatrix}_{rr'} (C^{-1})_{r',n'm'} = D^{\frac{1}{2}}(R)_{nn'} \times D^{j}(R)_{mm'} \, . \tag{35}$$

and the matrices $Z_{\frac{1}{2}}$, Z_j, $Z_{j+\frac{1}{2}}$ and $Z_{j-\frac{1}{2}}$ specified by equations (15), (16) and (17) are related by

$$(C^{-1})_{r,nm} (Z_{\frac{1}{2}})_{nn'} \times (Z_j)_{mm'}\, C_{n'm',r'} (Z^{-1}_{j+\frac{1}{2}} + Z^{-1}_{j-\frac{1}{2}})_{r'r''} = I_{rr''} \; .$$

PROOF

Taking the compex conjugate of equation (35) and noting that $C_{nm,r}$ is by convention real (cf. Cornwell[5]) we deduce that

$$C_{nm,r}\begin{bmatrix} D^{j+½}(R)^{*} & 0 \\ 0 & D^{j-½}(R)^{*}\end{bmatrix}_{rr'}(C^{-1})_{r',n'm'} = D^{½}(R)^{*}_{nn'} \times D^{j}(R)^{*}_{mm'} .$$

Then using equation (15) we obtain

$$C_{nm,r}\begin{bmatrix} Z^{-1}_{j+½}D^{j+½}(R)Z_{j+½} & 0 \\ 0 & Z^{-1}_{j-½}D^{j-½}(R)Z_{j-½}\end{bmatrix}(C^{-1})_{r',n'm'}$$

$$= (Z^{-1}_{½}D^{½}(R)Z_{½})_{nn'} \times (Z^{-1}_{j}D^{j}(R)Z_{j})_{mm'}$$

$$= ((Z^{-1}_{½}\times Z^{-1}_{j})(D^{½}(R)\times D^{j}(R))(Z_{½}\times Z_{j}))_{nm,n'm'}$$

$$= ((Z^{-1}_{½}\times Z^{-1}_{j})C\begin{bmatrix} D^{j+½}(R) & 0 \\ 0 & D^{j-½}(R)\end{bmatrix}(C^{-1}(Z_{½}\times Z_{j}))_{nm,n'm'} .$$

Thus the matrix

$$(C^{-1})_{r,nm}(Z_{½})_{nn'}\times(Z_{j})_{mm'}C_{n'm',r'}(Z^{-1}_{j+½}+Z^{-1}_{j-½})_{r'r''}$$

commutes with $\begin{bmatrix} D^{j+½}(R) & 0 \\ 0 & D^{j-½}(R)\end{bmatrix}$ for all $\mathbb{R}$.

Hence Schur's Lemma implies that it must be of the form

$$\begin{bmatrix} \sigma I_{2j+2} & 0 \\ 0 & \sigma' I_{2j}\end{bmatrix} \quad \text{with } \sigma,\ \sigma' \in \mathbb{C}.$$

Then, since the multiplicative constants in $Z_{½}$, Z_{j} and $Z_{j+½}$ may be chosen such that $\sigma = \sigma' = 1$, the result follows.

Using this Lemma we obtain for the creation operators:

$$\left\{\frac{\hbar}{Mc}\right\}^{½}[Q_{Rn}, \beta^{L}_{p,j,m}] \tag{36a}$$
$$= \frac{-i}{Mc}(\sigma_{\mu}P_{\mu})_{nn'}\left\{\begin{pmatrix} ½ & j & j+½ \\ n' & m & m+n'\end{pmatrix}\beta^{L}_{p,j+½,m+n'} + \begin{pmatrix} ½ & j & j-½ \\ n' & m & m+n'\end{pmatrix}\beta^{L}_{p,j-½,m+n'}\right\} ,$$

$$\left\{\frac{\hbar}{Mc}\right\}^{½}[Q_{Rn}, \beta^{L}_{p,j+½,m+½}] = \frac{-1}{Mc}(\sigma_{\mu}\sigma_{2}P_{\mu})_{nn'}\begin{pmatrix} ½ & j & j+½ \\ n' & m+½-n' & m+½\end{pmatrix}\beta^{L}_{\tilde{p},j,m+½-n'} \tag{36b}$$

$$\left\{\frac{\hbar}{Mc}\right\}^{½}[Q_{Rn}, \beta^{L}_{p,j-½,m-½}] = \frac{-1}{Mc}(\sigma_{\mu}\sigma_{2}P_{\mu})_{nn'}\begin{pmatrix} ½ & j & j-½ \\ n' & m-½-n' & m-½\end{pmatrix}\beta^{L}_{\tilde{p},j,m-½-n'} \tag{36c}$$

$$\left\{\frac{\hbar}{Mc}\right\}^{½}[Q_{Rn}, \beta^{L}_{\tilde{p},j,m}] = 0 \quad , \tag{36d}$$

$$\left\{\frac{\hbar}{Mc}\right\}^{\frac12} [\, Q_{Ln} \,, \beta^{L}_{p,j,m} \,] = 0 \quad , \tag{36d}$$

$$\left\{\frac{\hbar}{Mc}\right\}^{\frac12} [\, Q_{Ln} \,, \beta^{L}_{p,j+\frac12,m+\frac12}] = (\sigma_2)_{nn'} \left(\begin{array}{cc|c} \frac12 & j & j+\frac12 \\ n' & m+\frac12-n' & m+\frac12 \end{array}\right) \beta^{L}_{p,j,m+\frac12-n'} \quad , \tag{36f}$$

$$\left\{\frac{\hbar}{Mc}\right\}^{\frac12} [\, Q_{Ln} \,, \beta^{L}_{p,j-\frac12,m-\frac12}] = (\sigma_2)_{nn'} \left(\begin{array}{cc|c} \frac12 & j & j-\frac12 \\ n' & m-\frac12-n' & m-\frac12 \end{array}\right) \beta^{L}_{p,j,m-\frac12-n'} \quad , \tag{36g}$$

and

$$\left\{\frac{\hbar}{Mc}\right\}^{\frac12} [\, Q_{Ln} \,, \underset{\sim}{\beta}^{L}_{p,j,m} \,]$$
$$= i \left(\begin{array}{cc|c} \frac12 & j & j+\frac12 \\ n & m & m+n \end{array}\right) \beta^{L}_{p,j+\frac12,m+n} + i \left(\begin{array}{cc|c} \frac12 & j & j-\frac12 \\ n & m & m+n \end{array}\right) \beta^{L}_{p,j-\frac12,m+n} \quad . \tag{36h}$$

Similarly for the annihilation operators we obtain:

$$\left\{\frac{\hbar}{Mc}\right\}^{\frac12} [\, Q_{Rn} \,, \underset{\sim}{\alpha}^{L}_{p,j,m} \,]$$
$$= \frac{-1}{Mc}(\sigma_\mu p_\mu)_{nn'} \left\{ \left(\begin{array}{cc|c} \frac12 & j & j+\frac12 \\ n' & m & m+n' \end{array}\right) \alpha^{L}_{p,j+\frac12,m+n'} + \left(\begin{array}{cc|c} \frac12 & j & j-\frac12 \\ n' & m & m+n' \end{array}\right) \alpha^{L}_{p,j-\frac12,m+n'} \right\} , \tag{37a}$$

$$\left\{\frac{\hbar}{Mc}\right\}^{\frac12} [\, Q_{Rn} \,, \alpha^{L}_{p,j+\frac12,m+\frac12}]$$
$$= (-1)^{2j+1} \frac{1}{Mc}(\sigma_\mu \sigma_2 p_\mu)_{nn'} \left(\begin{array}{cc|c} \frac12 & j & j+\frac12 \\ n' & m+\frac12-n' & m+\frac12 \end{array}\right) \alpha^{L}_{p,j,m+\frac12-n'} \tag{37b}$$

$$\left\{\frac{\hbar}{Mc}\right\}^{\frac12} [\, Q_{Rn} \,, \alpha^{L}_{p,j-\frac12,m-\frac12}]$$
$$= (-1)^{2j+1} \frac{1}{Mc}(\sigma_\mu \sigma_2 p_\mu)_{nn'} \left(\begin{array}{cc|c} \frac12 & j & j-\frac12 \\ n' & m-\frac12-n' & m-\frac12 \end{array}\right) \alpha^{L}_{p,j,m-\frac12-n'} \tag{37c}$$

$$\left\{\frac{\hbar}{Mc}\right\}^{\frac12} [\, Q_{Rn} \,, \alpha^{L}_{p,j,m} \,] = 0 \quad , \tag{37d}$$

$$\left\{\frac{\hbar}{Mc}\right\}^{\frac12} [\, Q_{Ln} \,, \underset{\sim}{\alpha}^{L}_{p,j,m} \,] = 0 \quad , \tag{37e}$$

$$\left\{\frac{\hbar}{Mc}\right\}^{\frac12} [\, Q_{Ln} \,, \alpha^{L}_{p,j+\frac12,m+\frac12}]$$
$$= (-1)^{2j+1} (i\sigma_2)_{nn'} \left(\begin{array}{cc|c} \frac12 & j & j+\frac12 \\ n' & m+\frac12-n' & m+\frac12 \end{array}\right) \underset{\sim}{\alpha}^{L}_{p,j,m+\frac12-n'} \quad , \tag{37f}$$

$$\left\{\frac{\hbar}{Mc}\right\}^{\frac12} [\, Q_{Ln} \,, \alpha^{L}_{p,j-\frac12,m-\frac12}]$$
$$= (-1)^{2j+1} (i\sigma_2)_{nn'} \left(\begin{array}{cc|c} \frac12 & j & j-\frac12 \\ n' & m-\frac12-n' & m-\frac12 \end{array}\right) \underset{\sim}{\alpha}^{L}_{p,j,m-\frac12-n'} \quad , \tag{37g}$$

and

$$\left\{\frac{\hbar}{Mc}\right\}^{\frac12} [\, Q_{Ln} \,, \alpha^{L}_{p,j,m} \,]$$
$$= (-1)^{2j} \left\{ \left(\begin{array}{cc|c} \frac12 & j & j+\frac12 \\ n & m & m+n \end{array}\right) \alpha^{L}_{p,j+\frac12,m+n} + \left(\begin{array}{cc|c} \frac12 & j & j-\frac12 \\ n & m & m+n \end{array}\right) \alpha^{L}_{p,j-\frac12,m+n} \right\} . \tag{37h}$$

We can now construct the fields that form the left handed supermultiplet (the phase factors of equation (23) have been chosen with some foreknowledge). We define:

$$\chi^L_{j,m}(x)= \left\{\frac{1}{2\pi\hbar}\right\}^3\int dp^3\,\frac{Mc}{2p_4}\,\{(-1)^{2j}\rho\alpha^L_{p,j,m}\,e^{\frac{-i}{\hbar}p.x} - i\beta^L_{\tilde{p},j,m}\,e^{\frac{i}{\hbar}p.x}\} \qquad (38a)$$

$$\chi^L_{j+\frac{1}{2},m}(x) = \left\{\frac{1}{2\pi\hbar}\right\}^3\int dp^3\,\frac{Mc}{2p_4}\,\{\alpha^L_{p,j+\frac{1}{2},m}\,e^{\frac{-i}{\hbar}p.x} + \beta^L_{p,j+\frac{1}{2},m}\,e^{\frac{i}{\hbar}p.x}\} \qquad (38b)$$

$$\chi^L_{j-\frac{1}{2},m}(x) = \left\{\frac{1}{2\pi\hbar}\right\}^3\int dp^3\,\frac{Mc}{2p_4}\,\{\alpha^L_{p,j-\frac{1}{2},m}\,e^{\frac{-i}{\hbar}p.x} + \beta^L_{p,j-\frac{1}{2},m}\,e^{\frac{i}{\hbar}p.x}\} \qquad (38c)$$

and $$\mathcal{X}^L_{j,m}(x)= \left\{\frac{1}{2\pi\hbar}\right\}^3\int dp^3\,\frac{Mc}{2p_4}\,\{i(-1)^{2j}\alpha^L_{p,j,m}\,e^{\frac{-i}{\hbar}p.x} + \beta^L_{p,j,m}\,e^{\frac{i}{\hbar}p.x}\} \qquad (38d)$$

The action of the supersymmetry generators on these fields can now be evaluated to obtain:

$$[Q_{Rn},\mathcal{X}^L_{j,m}] = \left\{\frac{\hbar}{Mc}\right\}^{\frac{1}{2}}(\sigma_\mu\partial_\mu)_{nn'}\left\{\begin{pmatrix}\frac{1}{2} & j\\ n' & m\end{pmatrix}\middle|\begin{matrix}j+\frac{1}{2}\\ m+n'\end{matrix}\right)\chi^L_{j+\frac{1}{2},m+n'} + \begin{pmatrix}\frac{1}{2} & j\\ n' & m\end{pmatrix}\middle|\begin{matrix}j-\frac{1}{2}\\ m+n'\end{matrix}\Big)\chi^L_{j-\frac{1}{2},m+n'}\right\}, \qquad (39a)$$

$$[Q_{Rn},\chi^L_{j+\frac{1}{2},m+\frac{1}{2}}] = \left\{\frac{\hbar}{Mc}\right\}^{\frac{1}{2}}(\sigma_\mu\sigma_2\partial_\mu)_{nn'}\left(\begin{matrix}\frac{1}{2} & j\\ n' & m+\frac{1}{2}-n'\end{matrix}\middle|\begin{matrix}j+\frac{1}{2}\\ m+n'\end{matrix}\right)\chi^L_{j,m+\frac{1}{2}+n'}, \qquad (39b)$$

$$[Q_{Rn},\chi^L_{j-\frac{1}{2},m-\frac{1}{2}}] = \left\{\frac{\hbar}{Mc}\right\}^{\frac{1}{2}}(\sigma_\mu\sigma_2\partial_\mu)_{nn'}\left(\begin{matrix}\frac{1}{2} & j\\ n' & m-\frac{1}{2}-n'\end{matrix}\middle|\begin{matrix}j-\frac{1}{2}\\ m+n'\end{matrix}\right)\chi^L_{j,m-\frac{1}{2}+n'}, \qquad (39c)$$

$$[Q_{Rn},\chi^L_{j,m}] = 0 \qquad (39d)$$

$$[Q_{Ln},\chi^L_{j,m}] = \left\{\frac{Mc}{\hbar}\right\}^{\frac{1}{2}}\left\{\left(\begin{matrix}\frac{1}{2} & j\\ n & m\end{matrix}\middle|\begin{matrix}j+\frac{1}{2}\\ m+n\end{matrix}\right)\chi^L_{j+\frac{1}{2},m+n} + \left(\begin{matrix}\frac{1}{2} & j\\ n & m\end{matrix}\middle|\begin{matrix}j-\frac{1}{2}\\ m+n\end{matrix}\right)\chi^L_{j-\frac{1}{2},m+n}\right\} \qquad (39e)$$

$$[Q_{Ln},\chi^L_{j+\frac{1}{2},m+\frac{1}{2}}] = -i\left\{\frac{Mc}{\hbar}\right\}^{\frac{1}{2}}(\sigma_2)_{nn'}\left(\begin{matrix}\frac{1}{2} & j\\ n' & m+\frac{1}{2}-n'\end{matrix}\middle|\begin{matrix}j+\frac{1}{2}\\ m+n'\end{matrix}\right)\mathcal{X}^L_{j,m+\frac{1}{2}+n'}, \qquad (39f)$$

$$[Q_{Ln},\chi^L_{j-\frac{1}{2},m+\frac{1}{2}}] = -i\left\{\frac{Mc}{\hbar}\right\}^{\frac{1}{2}}(\sigma_2)_{nn'}\left(\begin{matrix}\frac{1}{2} & j\\ n' & m-\frac{1}{2}-n'\end{matrix}\middle|\begin{matrix}j-\frac{1}{2}\\ m+n'\end{matrix}\right)\mathcal{X}^L_{j,m-\frac{1}{2}+n'}, \qquad (39g)$$

and $[Q_{Ln},\mathcal{X}^L_{j,m}] = 0$. (39h)

This is not the most convenient way of expressing these relationships for the sequel. We define the field $\chi^L_{\frac{1}{2},n;j,m}$ by

$$\chi^L_{\frac{1}{2},n;j,m} = C_{nm,r}\,(\chi^L_{j+\frac{1}{2},a},\chi^L_{j-\frac{1}{2},b})_r \qquad (40)$$

with $a=1,2,\ldots,2j+2$ and $b=2j+3,\ldots,r$.

Then

$$U([\Lambda|t])\chi^L_{\frac{1}{2},n;j,m}(x)\; U([\Lambda|t])^{-1}$$
$$= \Gamma^{0,\frac{1}{2}}([\Lambda^{-1}|0])_{nn'}\Gamma^{0,j}([\Lambda^{-1}|0])_{mm'}\chi^L_{\frac{1}{2},n';j,m'}(\Lambda x+t) \;.$$

The final result of this section can now be given as a theorem.

THEOREM IV.2

Let the fields $\chi^L_{j,m}(x), \chi^L_{\frac{1}{2},n;j,m}(x)$ and $\mathcal{X}^L_{j,m}(x)$ be as defined by equations (38) and (40). Then the action of the supersymmetry generators Q_{Ln}, Q_{Rn} on these fields is given by:

$$[Q_{Ln}, \chi^L_{j,m}(x)] = \left\{\frac{Mc}{\hbar}\right\}^{\frac{1}{2}} \chi^L_{\frac{1}{2},n;j,m}(x)$$

$$[Q_{Ln}, \chi^L_{\frac{1}{2},r;j,m}(x)] = -\left\{\frac{Mc}{\hbar}\right\}^{\frac{1}{2}} (\sigma^L_2)_{nr}\, \mathcal{X}^L_{j,m}(x)$$

$$[Q_{Ln}, \mathcal{X}^L_{j,m}(x)] = 0$$

$$[Q_{Rn}, \chi^L_{j,m}(x)] = 0$$

$$[Q_{Rn}, \chi^L_{\frac{1}{2},r;j,m}(x)] = -\left\{\frac{\hbar}{Mc}\right\}^{\frac{1}{2}} (\sigma^L_\mu \sigma^L_2)_{nr}\partial_\mu\, \chi^L_{j,m}(x)$$

and

$$[Q_{Rn}, \mathcal{X}^L_{j,m}(x)] = -\left\{\frac{\hbar}{Mc}\right\}^{\frac{1}{2}} (\sigma^L_\mu)_{nr}\partial_\mu\, \chi^L_{\frac{1}{2},r;j,m}(x)$$

With $n,r=\frac{1}{2},-\frac{1}{2}$; $m=j,j-1,\ldots,-j+1,-j$ and j taking any integer or half integer value. We call these the left handed chiral supermultiplets.

PROOF

This is by straightforward algebra, so we give no details.

V. CONSTRUCTION OF RIGHT HANDED SUPERMULTIPLETS FROM LEFT HANDED SUPERMULTIPLETS

We could repeat the analysis of Section VI and thus construct a set of right handed fields. It is much simpler to construct one from the other. We first observe that if $\{A_i^{(j)},\ iA_i^{(j)},\ i = 1,2,3\}$ are the generators of $\Gamma^{0,j}([\Lambda|0])$ with the $A_i^{(j)}$ antihermitian and generating rotations, and $iA_i^{(j)}$ generating boosts, then $\{A_i^{(j)}, -iA_i^{(j)}\}$ are the corresponding generators of $\Gamma^{j,0}([\Lambda|0])$.

Consider the action of the matrix Z_j as specified in equations (15), (16) and (17) on a Lorentz boost, one obtains

$$Z_j^{-1}\Gamma^{0,j}([B(p,\hat{p})|0])Z_j = \Gamma^{j,0}([B(p,\hat{p})|0])^* \quad ,$$

so that

$$Z_j^{-1}\Gamma^{0,j}([\Lambda|0])Z_j = \Gamma^{j,0}([\Lambda|0])^* \quad , \tag{41}$$

for all $[\Lambda|0]$.

Now let $\psi_\alpha^L(x)$ be any second quantized field that transforms as

$$U([\Lambda|t])\psi_\alpha^L(x)U([\Lambda|t])^{-1} = \Gamma^{0,j}([\Lambda^{-1}|0])_{\alpha\beta}\psi_\beta^L(\Lambda x+t)$$

and consider the field

$$(Z_j)_{\alpha\alpha'}(\psi_{\alpha'}^L(x))^* = \psi_\alpha^R(x) \quad . \tag{42}$$

This then transforms as

$$U([\Lambda|t])\psi_\alpha^R(x)U([\Lambda|t])^{-1} = \Gamma^{j,0}([\Lambda^{-1}|0])_{\alpha\beta}\psi_\beta^R(\Lambda x+t)$$

We note that $\psi_\alpha^R(x)$ is constructed from the adjoints of the operators used for $\psi_\alpha^L(x)$ so that it can be considered as the antiparticle field of $\psi_\alpha^L(x)$. Also

$$(Z_j)_{\alpha\alpha'}(\psi_{\alpha'}^R(x))^* = (Z_jZ_j^*)_{\alpha\alpha'}\psi_\alpha^L(x) = (-1)^{2j}\psi_\alpha^L(x) \quad , \tag{43}$$

so that applying the transformation twice does not return us to the starting point, but produces an overall phase factor.

Now using equation (42) we can construct a set of right handed fields from the left handed fields defined in equations (38). We obtain

$$\chi'^{R}_{j,m}(x) = \left\{\frac{1}{2\pi\hbar}\right\}^3 \int dp^3 \, \frac{Mc}{2p_4} \left\{-i(-1)^{2j}\rho\alpha^R_{p,j,m} e^{\frac{-i}{\hbar}p.x} - (-1)^{2j}\beta^R_{\sim p,j,m} e^{\frac{i}{\hbar}p.x}\right\}, \quad (44a)$$

$$\chi^R_{j+\frac{1}{2},m}(x) = \left\{\frac{1}{2\pi\hbar}\right\}^3 \int dp^3 \, \frac{Mc}{2p_4} \left\{-(-1)^{2j}\alpha^R_{p,j+\frac{1}{2},m} e^{\frac{-i}{\hbar}p.x} + \beta^R_{p,j+\frac{1}{2},m} e^{\frac{i}{\hbar}p.x}\right\}, \quad (44b)$$

$$\chi^R_{j-\frac{1}{2},m}(x) = \left\{\frac{1}{2\pi\hbar}\right\}^3 \int dp^3 \, \frac{Mc}{2p_4} \left\{-(-1)^{2j}\alpha^R_{p,j-\frac{1}{2},m} e^{\frac{-i}{\hbar}p.x} + \beta_{p,j-\frac{1}{2},m} e^{\frac{i}{\hbar}p.x}\right\} \quad (44c)$$

and $$\chi'^{R}_{j,m}(x) = \left\{\frac{1}{2\pi\hbar}\right\}^3 \int dp^3 \, \frac{Mc}{2p_4} \left\{(-1)^{2j}\alpha_{\sim p,j,m} e^{\frac{-i}{\hbar}p.x} + i(-1)^{2j}\beta^R_{p,j,m} e^{\frac{i}{\hbar}p.x}\right\}, \quad (44d)$$

with $$\alpha^R_{p,k,m} = \left\{\frac{2p_4}{Mc}\right\}^{\frac{1}{2}} \Gamma^{k,0}([B(p,\hat{p})|0])_{mm'} \, b_{p,k,m'} \quad (45a)$$

and $$\beta^R_{p,k,m} = \left\{\frac{2p_4}{Mc}\right\}^{\frac{1}{2}} \Gamma^{k,0}([B(p,p)|0]Z_j)_{mm'} \, a^\dagger_{p,k,m'} \quad (45b)$$

for $k=j, j+\frac{1}{2}, j-\frac{1}{2}$ and, of course, similar definitions for $\alpha^R_{\sim p,j,m}$ and $\beta^R_{\sim p,j,m}$. It is convenient to work with a slightly different set of fields. We define

$$\chi^R_{j,m}(x) = (-1)^{2j}\chi'^{R}_{j,m}(x) \quad (46)$$

and $$\chi^R_{j,m}(x) = (-1)^{2j}\chi'^{R}_{j,m}(x) \quad . \quad (47)$$

The action of the supersymmetry generators on a right handed field is given by

$$[Q_{Ln}, \chi^R_{j,m}(x)] = (Z_j)_{mm'}(i\sigma_2)_{nn'}(-1)^{2j}([Q_{Rn'}, \chi^L_{j,m'}(x)])^* \quad (48a)$$

and $$[Q_{Rn}, \chi^R_{j,m}(x)] = (Z_j)_{mm'}(i\sigma_2)_{nn'}(-1)^{2j-1}([Q_{Ln'}, \chi^L_{j,m'}(x)])^*. \quad (48b)$$

Thus for the field $\chi^R_{j,m}(x)$ as defined above

$$[Q_{Ln}, \chi^R_{j,m}] = \left\{\frac{\hbar}{Mc}\right\}^{\frac{1}{2}} (\sigma^R_\mu \partial_\mu)_{nn'} \left\{ \begin{pmatrix} \frac{1}{2} & j & j+\frac{1}{2} \\ n' & m & m+n' \end{pmatrix} \chi^R_{j+\frac{1}{2},m+n'} + \begin{pmatrix} \frac{1}{2} & j & j-\frac{1}{2} \\ n' & m & m+n' \end{pmatrix} \chi^R_{j-\frac{1}{2},m+n'} \right\} \quad . \quad (49a)$$

Similarly

$$[Q_{Ln}, \chi^R_{j+\frac{1}{2},m+\frac{1}{2}}] = i\left\{\frac{\hbar}{Mc}\right\}^{\frac{1}{2}} (\sigma^R_\mu \sigma^R_2 \partial_\mu)_{nn'} \begin{pmatrix} \frac{1}{2} & j & j+\frac{1}{2} \\ n' & m+\frac{1}{2}-n' & m+n' \end{pmatrix} \chi^R_{j,m+\frac{1}{2}+n''} \quad (49b)$$

$$[\,Q_{Ln}\,,\chi^R_{j-\frac12,\,m+\frac12}] = i\left\{\frac{\hbar}{Mc}\right\}^{\frac12}(\sigma^R_\mu\sigma^R_2\partial_\mu\,)_{nn'}\begin{pmatrix}\frac12 & j\\ n', & m-\frac12-n'\end{pmatrix}\!\!\left|\begin{matrix}j-\frac12\\ m+n'\end{matrix}\right)\chi^R_{j,\,m-\frac12+n''} \tag{49c}$$

$$[\,Q_{Ln}\,,\chi^R_{j,m}\,] = 0 \quad , \tag{49d}$$

$$[\,Q_{Rn}\,,\chi^R_{j,m}\,] = i\left\{\frac{Mc}{\hbar}\right\}^{\frac12}(\sigma_2)_{nn'}\left\{\begin{pmatrix}\frac12 & j\\ n', & m\end{pmatrix}\!\!\left|\begin{matrix}j+\frac12\\ m+n'\end{matrix}\right)\chi^R_{j+\frac12,\,m+n'} + \begin{pmatrix}\frac12 & j\\ n', & m\end{pmatrix}\!\!\left|\begin{matrix}j-\frac12\\ m+n'\end{matrix}\right)\chi^R_{j-\frac12,\,m+n'}\right\} \quad , \tag{49e}$$

$$[\,Q_{Rn}\,,\chi^R_{j+\frac12,\,m+\frac12}] = \left\{\frac{Mc}{\hbar}\right\}^{\frac12}\begin{pmatrix}\frac12 & j\\ n', & m+\frac12-n'\end{pmatrix}\!\!\left|\begin{matrix}j+\frac12\\ m+n'\end{matrix}\right)\dot{\chi}^R_{j,\,m+\frac12+n'} \quad , \tag{49f}$$

$$[\,Q_{Rn}\,,\chi^R_{j-\frac12,\,m-\frac12}] = \left\{\frac{Mc}{\hbar}\right\}^{\frac12}\begin{pmatrix}\frac12 & j\\ n', & m-\frac12-n'\end{pmatrix}\!\!\left|\begin{matrix}j-\frac12\\ m+n'\end{matrix}\right)\dot{\chi}^R_{j,\,m-\frac12+n'} \quad , \tag{49g}$$

and $[\,Q_{Rn}\,,\dot{\chi}^R_{j,m}\,] = 0$. (49h)

As with the left handed fields it is convenient to construct a single field from the $(j+\frac12)$ and $(j-\frac12)$ fields ie.

$$\chi^R_{\frac12,n;j,m} = C_{nm,r}\,(\chi^R_{j+\frac12,a}\,,\chi^R_{j-\frac12,b}\,)_r \quad . \tag{50}$$

It is then a matter of straightforward algebra to prove the following theorem. Note that we have written this theorem in terms of the right handed Pauli matrices as defined in Appendix A.

THEOREM V.1

Let the fields $\chi^R_{j,m}(x)$, $\chi^R_{\frac12,n;j,m}(x)$ and $\dot{\chi}^R_{j,m}(x)$ be as defined by equations (44), (46), (47) and (50). Then the action of the supersymmetry generators on these fields is given by:

$$[\,Q_{Rn},\chi^R_{j,m}(x)] = i\left\{\frac{Mc}{\hbar}\right\}^{\frac12}\chi^R_{\frac12,n;j,m}(x) \quad ,$$

$$[\,Q_{Rn},\chi^R_{\frac12,r;j,m}(x)] = -i\left\{\frac{Mc}{\hbar}\right\}^{\frac12}(\sigma^R_2)_{nr}\,\dot{\chi}^R_{j,m}(x) \quad ,$$

$$[\,Q_{Rn},\dot{\chi}^R_{j,m}(x)] = 0 \quad ,$$

$$[\,Q_{Ln},\chi^R_{j,m}(x)] = 0 \quad ,$$

$$[\,Q_{Rn},\chi^R_{\frac12,r;j,m}(x)] = i\left\{\frac{\hbar}{Mc}\right\}^{\frac12}(\sigma^R_\mu\sigma^R_2)_{nr}\partial_\mu\,\chi^R_{j,m}(x)$$

and
$$[\,Q_{Ln},\dot{\chi}^R_{j,m}(x)] = i\left\{\frac{\hbar}{Mc}\right\}^{\frac12}(\sigma^R_\mu)_{nr}\partial_\mu\,\chi^R_{\frac12,r;j,m}(x) \quad .$$

VI. SUPERMULTIPLETS THAT ARE SYMMETRIC UNDER THE INTERCHANGE OF L AND R.

While the fields used in the previous two sections are perfectly adequate as they stand, it is more convenient, and more pleasing, to redefine them so that the commutators or anticommutators of Theorems IV. and V. become symmetric under the interchange of L and R. We achieve this by altering the phase factors of the fields. This leaves an overall phase factor undetermined. To reduce the choice we demand that the differential operator linking the fields is also symmetric in the interchange of L and R. This still does not give a unique choice. Our choice is such that the Dirac equation takes its standard form. We define:

$$\chi'^{R}_{j,m} = e^{-ib\pi}\chi^{R}_{j,m}\ , \tag{51a}$$

$$\chi'^{L}_{j,m} = e^{ib\pi}\chi^{L}_{j,m}\ , \tag{51b}$$

$$\mathcal{X}'^{R}_{j,m} = e^{-ic\pi}\mathcal{X}^{R}_{j,m}\ , \tag{51c}$$

$$\mathcal{X}'^{L}_{j,m} = e^{ic\pi}\mathcal{X}^{L}_{j,m}\ , \tag{51d}$$

$$\chi^{R}_{½,n;j,\bar{m}} = e^{-ia\pi}\chi^{R}_{½,n;j,m} \tag{51e}$$

and
$$\chi^{L}_{½,n;j,\bar{m}} = e^{ia\pi}\chi^{L}_{½,n;j,m} \tag{51f}$$

with $a,b,c \in \mathbb{R}$. Then using the results of Theorems (IV.2) and (V.1) we find that $c = a+1/4$ and $b = a-3/4$. Thus we can choose (say) $a\epsilon[0,2\pi]$ and the parameters b,c are then fixed.

To restrict this choice we require that if $\chi^{L}_{½,n;j,m}(x)$ and $\chi^{R}_{½,n;j,m}(x)$ are self conjugate for $j=0$, that is, in terms of the creation and annihilation operators we put $a_{p,j,m} = b_{p,j,m}$, then the differential operators linking the fields are also symmetric in the interchange of L and R. Since for $j\neq 0$ the differential operator acting on the index m can be constructed from the $j=½$ operator by a sequence of tensor products this will be true for all j. We find that

$$(\sigma^{L}_{\mu}\partial_{\mu})_{nn'}\chi^{L}_{½,n';0,0}(x) = \frac{iMc}{\hbar}\chi^{R}_{½,n;0,0}(x) \tag{52a}$$

and $$(\sigma^R_\mu\partial_\mu)_{nn'}\chi^R_{½,n';0,0}(x) = \frac{iMc}{\hbar}\chi^L_{½,n;0,0}(x) \quad . \tag{52b}$$

The restriction then becomes $a=n/2$, $n=0,\pm1,\pm2,\ldots$. Our choice is $a=-1/2$ so that $b=-5/4$ and $c=-1/4$.

THEOREM VI.1

With the above choices the action of the supersymmetry generators on the fields $\{\chi^R_{j,m}(x),\ \chi'^R_{½,n;j,m}(x),\ \chi'^R_{j,m}(x)\}$ and $\{\chi^L_{j,m}(x),\ \chi'^L_{½,n;j,m}(x),\ \chi'^L_{j,m}(x)\}$ of the chiral supermultiplets can be written:

$$[Q_{Rn},\chi'^R_{j,m}(x)] = e^{i3\pi/4}\left\{\frac{Mc}{\hbar}\right\}^{½}\chi'^R_{½,n;j,m}(x) \quad , \tag{53a}$$

$$[Q_{Rn},\chi'^R_{½,r;j,m}(x)] = e^{-i3\pi/4}\left\{\frac{Mc}{\hbar}\right\}^{½}(\sigma^R_2)_{nr}\chi^R_{j,m}(x) \quad , \tag{53b}$$

$$[Q_{Rn},\chi^R_{j,m}(x)] = 0 \quad , \tag{53c}$$

$$[Q_{Ln},\chi'^R_{j,m}(x)] = 0 \quad , \tag{53d}$$

$$[Q_{Ln},\chi'^R_{½,r;j,m}(x)] = e^{i3\pi/4}\left\{\frac{\hbar}{Mc}\right\}^{½}(\sigma^R_\mu\sigma^R_2\partial_\mu)_{nr}\chi'^R_{j,m}(x) \quad , \tag{53e}$$

and $$[Q_{Ln},\chi^R_{j,m}(x)] = e^{i3\pi/4}\left\{\frac{\hbar}{Mc}\right\}^{½}(\sigma^R_\mu\partial_\mu)_{nr}\chi'^R_{½,r;j,m}(x) \quad ; \tag{53f}$$

$$[Q_{Rn},\chi'^L_{j,m}(x)] = e^{i3\pi/4}\left\{\frac{Mc}{\hbar}\right\}^{½}\chi'^L_{½,n;j,m}(x) \quad , \tag{54a}$$

$$[Q_{Rn},\chi'^L_{½,r;j,m}(x)] = e^{-i3\pi/4}\left\{\frac{Mc}{\hbar}\right\}^{½}(\sigma^L_2)_{nr}\chi^L_{j,m}(x) \quad , \tag{54b}$$

$$[Q_{Rn},\chi^L_{j,m}(x)] = 0 \quad , \tag{54c}$$

$$[Q_{Ln},\chi'^L_{j,m}(x)] = 0 \quad , \tag{54d}$$

$$[Q_{Ln},\chi'^L_{½,r;j,m}(x)] = e^{i3\pi/4}\left\{\frac{\hbar}{Mc}\right\}^{½}(\sigma^L_\mu\sigma^L_2\partial_\mu)_{nr}\chi'^L_{j,m}(x) \quad , \tag{54e}$$

and $$[Q_{Ln},\chi^L_{j,m}(x)] = e^{i3\pi/4}\left\{\frac{\hbar}{Mc}\right\}^{½}(\sigma^L_\mu\partial_\mu)_{nr}\chi'^L_{½,r;j,m}(x) \quad . \tag{54f}$$

These sets of equations are then symmetric in the interchange of *L* and *R*. Also the differential operators relating the fields are symmetric in the interchange of L and R if the fields are self conjugate.

This is the form we will use for these expressions from now on.

VII. SUPERMULTIPLETS OF FIELDS THAT SATISFY WAVE EQUATIONS.

In sections IV, V and VI we detailed the construction of chiral supermultiplets for both left handed and right handed fields. By choice we constructed the right handed fields as the antiparticle fields of the left handed fields, but we could have chosen to construct them from the same set of creation and annihilation operators. In all of these chiral supermultiplets the number of independent field components is the same as the number of independent creation (or annihilation) operators. That is, each field has $(2k+1)$ components, with $k=j, j+½$ and $j-½$ as appropriate. Such fields describe particles of a single spin value and obey only the Klein-Gordon equation.

In this section we want to examine supermultiplets of fields that do obey field equations other than the Klein-Gordon equation. This implies that not all the field components are linearly independent. In this discussion we will assume that $k\neq 0$, since the case $k=0$ is different and essentially trivial. The reason for introducing additional field components, according to Weinberg[3], is that requiring a field to transform in a simple way under P (parity), and C (charge conjugation) cannot be achieved with a (2k+1) component field. These fields transform in a simple way under T and CP but not under C or P. To obtain field that does transform simply under C and P it is convenient to use $2(2k+1)$ component fields that transform as $\Gamma^{0,k}([\Lambda|0]) + \Gamma^{k,0}([\Lambda|0])$. In the case of a field corresponding to a particle that is its own antiparticle, this would be constructed from $(2k+1)$ creation operators for each 4-momentum and $(2k+1)$ annihilation operators for each 4-momentum. For the distinct antiparticle case the number of operators is doubled.

We note that fields constructed in this way consist of particles of a single spin value. Many other field types are considered in the physics literature (eg. the vector field A_μ transforming as $\Lambda = S\,\Gamma^{½,½}(\Lambda)S^{-1}$ for some similarity transformation S). These consist of several spin values constrained in some way to remove unwanted components.

Given the left handed field $\psi^L_{k,m}$ we now demonstrate how Weinberg's analysis[3] can be extended to construct a right handed field $\psi^R_{k,m}$ so that the field

$$\psi_{k,\bar{\sigma}} = \begin{bmatrix} \psi^L_{k,m} \\ \psi^R_{k,m} \end{bmatrix} \tag{55}$$

satisfies a differential equation in addition to the Klein-Gordon equation. To do this we construct the differential operators relating the left and right handed parts of the field.

We have already given the result for $k=½$ in equation (52), which can be rewritten as the single equation

$$\frac{i\hbar}{Mc}\begin{bmatrix} 0 & (\sigma^R_\mu\partial_\mu) \\ (\sigma^L_\mu\partial_\mu) & 0 \end{bmatrix}_{\sigma\sigma'} \psi_{½,\sigma'} = \psi_{½,\sigma} \quad . \tag{56}$$

For all $k>0$ the differential equation will take this form and can be recognised as the Dirac equation. We define the operators $\Pi^L_k(\partial)$ and $\Pi^R_k(\partial)$ to be such that the field $\psi_{k,\sigma}$ satisfies

$$\begin{bmatrix} 0 & \Pi^R_k(\partial) \\ \Pi^L_k(\partial) & 0 \end{bmatrix}_{\sigma\sigma'} \psi_{k,\sigma'} = \psi_{k,\sigma} \quad . \tag{57}$$

The following proposition then enables us to evaluate these operators successively for each $k=1,3/2,\ldots$.

PROPOSITION VII.1

Suppose $\psi^A_{½,\sigma'}$, $\psi^A_{j-½,\sigma''}$, $\psi^A_{j,\sigma''}$ *and* $\psi^A_{j+½,\sigma'''}$ are a set of fields related by

$$C_{nm,r}(\psi^A_{j+½,\sigma'''}\,,\ \psi^A_{j-½,\sigma'})_r = \psi^A_{½,n} \times \psi^A_{j,m} \tag{58}$$

for $A=L$ or $A=R$ and $C_{nm,r}$ as defined by equation (6). Further suppose we know the differential operators $\Pi^{A}_{k}(\partial)$ for $A=L$ or $A=R$ and $k=½, j$. The differential operators relating the left and right $j+½$ and $j-½$ fields are then given by

$$\left(\Pi^{A}_{j+½}(\partial)\right)_{\sigma\rho} = \begin{pmatrix} ½ & j & j+½ \\ n & m & \sigma \end{pmatrix} (\Pi^{A}_{½}(\partial))_{nn'} \times (\Pi^{A}_{j}(\partial))_{mm'} \begin{pmatrix} ½ & j & j+½ \\ n' & m' & \rho \end{pmatrix}$$

and

$$\left(\Pi^{A}_{j-½}(\partial)\right)_{\sigma\rho} = \begin{pmatrix} ½ & j & j-½ \\ n & m & \sigma \end{pmatrix} (\Pi^{A}_{½}(\partial))_{nn'} \times (\Pi^{A}_{j}(\partial))_{mm'} \begin{pmatrix} ½ & j & j-½ \\ n' & m' & \rho \end{pmatrix}$$

PROOF

Consider equation (56) for $A=L$ to obtain

$$(\psi^{L}_{j+½,\sigma''}, \psi^{L}_{j-½,\sigma'})_{r}$$
$$= C^{-1}_{r,nm}(\Pi^{R}_{½}(\partial))_{nn'} \times (\Pi^{R}_{j}(\partial))_{mm'} C_{n'm',r'} (\psi^{R}_{j+½,\rho''}, \psi^{R}_{j-½,\rho'})_{r'} .$$

But $C_{nm,r}$ is the Clebsch-Gordan matrix. The results follow for A=L. By symmetry they are also true for $A=R$.

We observe that the degree of the differential operator $\Pi_{k}(\partial)$ is given by 2k so that we only get a first order equation if $k=½$.

Now consider the simplest supersymmetric model as originally given by Wess and Zumino[6]. This is obtained by putting $j=0$ in equations (53) and (54), noting that the $j-½$ field does not exist. (The models with $j\neq 0$ are a straightforward generalization of this case, as we will show below.) Let

$$X_{½,n;\alpha} = \begin{bmatrix} X^{L}_{½,n;0,0} \\ X^{R}_{½,n';0,0} \end{bmatrix}$$

and recall the definition of Q^{C}_{α} from Appendix B, then equations (53b), (53e), (54b) and (54e) can be combined to give

$$[Q^c_\alpha, \chi_\beta] = e^{-i3\pi/4}\left\{\frac{Mc}{\hbar}\right\}^{1/2}\begin{bmatrix}(\sigma_2^L)_{nn'}\chi^L_{0,0} & 0 \\ 0 & (\sigma_2^R)_{nn'}\chi^R_{0,0}\end{bmatrix}_{\alpha\beta}$$

$$+\ e^{i3\pi/4}\left\{\frac{\hbar}{Mc}\right\}^{1/2}\begin{bmatrix}0 & (\sigma^R_\mu\sigma^R_2\partial_\mu)_{nn'}\chi^R_{0,0} \\ (\sigma^L_\mu\sigma^L_2\partial_\mu)_{nn'}\chi^L_{0,0} & 0\end{bmatrix}_{\alpha\beta} .$$

Now define the scalar fields A, B, F and G by

$$\chi^L_{0,0} = \frac{1}{2^{1/2}}\ (F + iG) \quad ,$$

$$\chi^R_{0,0} = \frac{1}{2^{1/2}}\ (F - iG) \quad ,$$

$$\chi^L_{0,0} = \frac{1}{2^{1/2}}\ (A + iB)$$

and $$\chi^L_{0,0} = \frac{1}{2^{1/2}}\ (A - iB) \quad .$$

Then, noting the definitions of the chiral Dirac matrices in Appendix A we obtain

$$[Q^c_\alpha, \chi_\beta] = \frac{1}{2}\ e^{-i3\pi/4}\left\{\frac{Mc}{\hbar}\right\}^{1/2}\{(C^c)_{\alpha\beta}F + (\gamma^c_5 C^c)_{\alpha\beta}iG\}$$
$$-\ \frac{1}{2}\ e^{3\pi/4}\left\{\frac{\hbar}{Mc}\right\}^{1/2}\{(\gamma^{c\mu}C^c)_{\alpha\beta}\partial_\mu A - (\gamma^c_5\gamma^{c\mu}C^c)_{\alpha\beta}\partial_\mu iB\} \quad , \tag{59a}$$

$$[Q^c_\alpha, A] = \frac{1}{2}\ e^{i\pi/4}\left\{\frac{Mc}{\hbar}\right\}^{1/2}\chi_\alpha \quad , \tag{59b}$$

$$[Q^c_\alpha, iB] = \frac{1}{2}\ e^{i\pi/4}\left\{\frac{Mc}{\hbar}\right\}^{1/2}(\gamma^c_5)_{\alpha\beta}\ \chi_\beta \quad , \tag{59c}$$

$$[Q^c_\alpha, F] = -\ \frac{1}{2}\ e^{i3\pi/4}\left\{\frac{\hbar}{Mc}\right\}^{1/2}(\gamma^{c\mu})_{\alpha\beta}\partial_\mu\ \chi_\beta \quad , \tag{59d}$$

and $$[Q^c_\alpha, iG] = \frac{1}{2}\ e^{i3\pi/4}\left\{\frac{\hbar}{Mc}\right\}^{1/2}(\gamma^c_5\gamma^{c\mu})_{\alpha\beta}\partial_\mu\ \chi_\beta \quad . \tag{59e}$$

These expressions were obtained using the chiral representation of the Dirac matrices. Since we can transform to any representation with a similarity transformation we can delete the superscript c.

Now we want to examine the consequences of requiring that the spinor χ_α satisfies the Dirac equation, which, rewriting equation (56), is

$$(\gamma^\mu\partial_\mu)_{\alpha\beta}\chi_\beta = -i\ \frac{Mc}{\hbar}\ \chi_\alpha \quad .$$

Then acting on equation (59a) with this operator we obtain

$$[Q_\alpha, \chi_\gamma] = \frac{1}{2} e^{i3\pi/4} \left\{\frac{Mc}{\hbar}\right\}^{½} \{(\gamma^\mu C)_{\alpha\gamma}\partial_\mu F - (\gamma^\mu \gamma_5 C)_{\alpha\gamma}\partial_\mu iG\}$$

$$+ \frac{1}{2} e^{\pi/4} \left\{\frac{\hbar}{Mc}\right\}^{3/2} \{(\gamma^\mu \gamma^\rho C)_{\alpha\gamma}\partial_\mu\partial_\rho A - (\gamma^\mu\gamma^\rho\gamma_5 C)_{\alpha\gamma}\partial_\mu\partial_\rho iB\} .$$

Now $(\gamma^\mu\gamma^\rho\partial_\mu\partial_\rho)_{\alpha\beta} = \delta_{\alpha\beta}\partial_\mu\partial^\mu$, and since all the fields must satisfy the Klein-Gordon equation $\partial^\mu\partial_\mu A = -\left\{\frac{Mc}{\hbar}\right\}^2 A$ and $\partial^\mu\partial_\mu B = -\left\{\frac{Mc}{\hbar}\right\}^2 B$, it follows that

$$[Q_\alpha, \chi_\beta] = \frac{1}{2} e^{i\pi/4} \left\{\frac{Mc}{\hbar}\right\}^{½} \{(C)_{\alpha\beta}A - (\gamma_5 C)_{\alpha\beta}B\}$$

$$+ \frac{1}{2} e^{3\pi/4} \left\{\frac{\hbar}{Mc}\right\}^{½} \{(\gamma^\mu C)_{\alpha\beta}\partial_\mu F + (\gamma_5\gamma^\mu C)_{\alpha\beta}\partial_\mu iG\} .$$

Comparing this with equation (59a) we see that if χ_β satisfies the Dirac equation then $A = -F$ and $B = G$. Similar arguments using equations (59b), (59c), (59d) and (59e) demonstrate that given the three constraints

$$(\gamma^\mu\partial_\mu)_{\alpha\beta}\chi_\beta = -i\,\frac{Mc}{\hbar}\,\chi_\alpha \quad , \quad A = -F \;\; and \;\; B = G$$

any one implies the other two. These considerations clarify the nature of the 'auxiliary fields' F and G.

To generalize to the case of $j \neq 0$ we observe that the spin index m is left unchanged by the action of the supersymmetry generators in equations (53) and (54). Hence using the differential operators $\Pi_j(\partial)$, we can construct supermultiplets obeying the commutators of these equations that are either left handed or right handed with respect to the spin index j. It follows that we can construct sets of fields generalising equations (59) that are either left handed or right handed. That is, we can obtain the following set of commutators.

$$[Q_\alpha, \chi^A_{\beta;j,m}] = \frac{1}{2} e^{-i3\pi/4} \left\{\frac{Mc}{\hbar}\right\}^{½} \{(C)_{\alpha\beta}F^A_{j,m} + (\gamma_5 C)_{\alpha\beta}iG^A_{j,m}\}$$

$$- \frac{1}{2} e^{3\pi/4} \left\{\frac{\hbar}{Mc}\right\}^{½} \{(\gamma^\mu C)_{\alpha\beta}\partial_\mu A^A_{j,m} - (\gamma_5\gamma^\mu C)_{\alpha\beta}\partial_\mu iB^A_{j,m}\} , \quad (60a)$$

$$[Q_\alpha, A^A_{j,m}] = \frac{1}{2} e^{i\pi/4} \left\{\frac{Mc}{\hbar}\right\}^{½} \chi^A_{\alpha;j,m} , \quad (60b)$$

$$[\,Q_\alpha\,,iB^A_{j,m}\,] = \frac{1}{2}\,e^{i\pi/4}\left\{\frac{Mc}{\hbar}\right\}^{½}(\gamma_5)_{\alpha\beta}\,\chi^A_{\beta;j,m}\quad, \tag{60c}$$

$$[\,Q_\alpha\,,F^A_{j,m}\,] = -\frac{1}{2}\,e^{i3\pi/4}\left\{\frac{\hbar}{Mc}\right\}^{½}(\gamma^\mu_5)_{\alpha\beta}\partial_\mu\,\chi^A_{\beta;j,m}\,, \tag{60d}$$

and $$[\,Q_\alpha\,,iG^A_{j,m}\,] = \frac{1}{2}\,e^{i3\pi/4}\left\{\frac{\hbar}{Mc}\right\}^{½}(\gamma_5\gamma^\mu)_{\alpha\beta}\partial_\mu\,\chi^A_{\beta;j,m}\quad, \tag{60e}$$

for $A = L$ or $A = R$. Then, as with the $j=0$ model, we obtain the three constraints, any one of which implies the other two ie.

$$(\gamma^\mu\partial_\mu)_{\alpha\beta}\,\chi^A_{\beta;j,m} = -i\,\frac{Mc}{\hbar}\,\chi^A_{\alpha;j,m},\quad A^A_{j,m} = -F^A_{j,m}\ \text{and}\ \ B^A_{j,m} = G^A_{j,m}.$$

Now suppose we require a combination of supermultiplets, given by equation (60), such that each field obeys some field equation other than the Klein-Gordon equation. The obvious choice is to combine a left handed and right handed supermultiplet and require that the fields are related as in Proposition VII.1. We then have the fields

$$A_{j,\beta} = \begin{bmatrix} A^L_{j,m} \\ A^R_{j,m} \end{bmatrix}_\beta \quad,$$

$$B_{j,\beta} = \begin{bmatrix} B^L_{j,m} \\ B^R_{j,m} \end{bmatrix}_\beta \quad,$$

$$F_{j,\beta} = \begin{bmatrix} F^L_{j,m} \\ F^R_{j,m} \end{bmatrix}_\beta \quad,$$

$$G_{j,\beta} = \begin{bmatrix} G^L_{j,m} \\ G^R_{j,m} \end{bmatrix}_\beta$$

and $$\chi_{\alpha;j,\beta} = \begin{bmatrix} \chi^L_{\alpha;j,m} \\ \chi^R_{\alpha;j,m} \end{bmatrix}_\beta$$

with $\beta = 1,2,\ldots,2(2j+1)$. Then, as before, we obtain the three constraints, any one of which implies the other two, ie.

$$(\gamma^\mu\partial_\mu)_{\alpha\beta}\,\chi_{\beta;j,\gamma} = -i\,\frac{Mc}{\hbar}\,\chi_{\alpha;j,\gamma}\,,\quad A_{j,\gamma} = -F_{j,\gamma}\ \text{and}\ \ B_{j,\gamma} = G_{j,\gamma}.$$

By construction these fields obey the equations

$$\begin{bmatrix} 0 & \Pi^R_j(\partial) \\ \Pi^L_j(\partial) & 0 \end{bmatrix}_{\sigma\sigma'} A_{j,\sigma'} = A_{j,\sigma}\quad, \tag{61a}$$

$$\begin{bmatrix} 0 & \Pi_j^R(\partial) \\ \Pi_j^L(\partial) & 0 \end{bmatrix}_{\sigma\sigma'} B_{j,\sigma'} = B_{j,\sigma} \quad , \tag{61b}$$

$$\begin{bmatrix} 0 & \Pi_j^R(\partial) \\ \Pi_j^L(\partial) & 0 \end{bmatrix}_{\sigma\sigma'} (\gamma^\mu \partial_\mu)_{\alpha\alpha'} \chi_{\alpha';j,\sigma'} = \frac{-iMc}{\hbar} \chi_{\alpha;j,\sigma} \ . \tag{61c}$$

APPENDIX A

Our metric is $g_{\mu\lambda} = diag(-1, -1, -1, 1)$. We use the standard definition for the Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and also } \sigma_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

A greek subscript on a Pauli matrix is assumed to take the values *1, 2, 3, 4* and a latin subscript the values *1, 2, 3.* Repeated subscripts are assumed to be summed.

We define the left and right handed Pauli matrices by:

$$\sigma^L_\mu = \sigma_\mu \qquad \text{for } \mu = 1,\ 2,\ 3,\ 4\ , \tag{A1}$$

and

$$\sigma^R_\mu = \begin{cases} \sigma_\mu & \text{for } \mu = 1,\ 2,\ 3 \\ \sigma_4 & \text{for } \mu = 4 \end{cases} \quad . \tag{A2}$$

Then $(\sigma^L_\mu)^2 = (\sigma^R_\mu)^2 = 1$ and $\sigma^L_\mu g_{\mu\lambda} = \sigma^R_\lambda$.

For the Dirac matrices we use the conventions of our previous paper[7] . The chiral Dirac matrices can be written in terms of the left and right handed Pauli matrices as:

$$\gamma^C_\mu = \begin{bmatrix} 0 & -\sigma^L_\mu \\ -\sigma^R_\mu & 0 \end{bmatrix}, \quad \sigma^C_{\lambda\mu} = \begin{bmatrix} \sigma^L_\lambda\sigma^R_\mu & 0 \\ 0 & \sigma^R_\mu\sigma^L_\lambda \end{bmatrix}$$

$$\gamma^C_\mu = \begin{bmatrix} \sigma^L_4 & 0 \\ 0 & -\sigma^R_4 \end{bmatrix}, \quad C^C = \begin{bmatrix} \sigma^L_2 & 0 \\ 0 & \sigma^R_2 \end{bmatrix} \quad . \tag{A3}$$

In section IV we make use of the following identities:

$$(\Gamma^{0,j}(\Lambda^{-1}))^\dagger = \Gamma^{j,0}(\Lambda) \quad , \tag{A4}$$

and with $p^\mu = \Lambda^\mu{}_\sigma(0,\ 0,\ 0,\ Mc)^\sigma$

$$\Gamma^{0,½}(\Lambda)_{nr'}\Gamma^{½,0}(\Lambda^{-1})_{r'r} = \frac{1}{Mc}(p^\mu\sigma^L_\mu)_{nr} \tag{A5}$$

and

$$\Gamma^{½,0}(\Lambda)_{nr'}\Gamma^{0,½}(\Lambda^{-1})_{r'r} = \frac{1}{Mc}(p^\mu\sigma^R_\mu)_{nr} \tag{A6}$$

APPENDIX B

We consider the 4-dimensional real super Poincaré algebra written in terms of the 6 Hermitian generators of the Lorentz algebra $M_{\lambda\mu}$, $\lambda,\mu = 1, 2, 3, 4$ and $M_{\lambda\mu} = -M_{\mu\lambda}$; the 4 Hermitian translation generators P_σ , $\sigma = 1, 2, 3, 4$ and the supersymmetry generators Q_α , $\alpha = 1, 2, 3, 4$. The algebra is then:

$$[M_{\lambda\mu}, M_{\sigma\rho}] = \hbar/i\{g_{\lambda\sigma}M_{\mu\rho} - g_{\lambda\rho}M_{\mu\sigma} - g_{\mu\sigma}M_{\lambda\rho} + g_{\mu\rho}M_{\lambda\sigma}\} \quad , \qquad \text{(B2a)}$$

$$[M_{\lambda\mu}, P_\sigma] = \hbar/i\{g_{\lambda\sigma}P_\mu - g_{\mu\sigma}P_\lambda\} \quad , \qquad \text{(B2b)}$$

$$[P_\sigma , P_\rho] = 0 \quad , \qquad \text{(B2c)}$$

$$[M_{\lambda\mu}, Q_\alpha] = \hbar/i(\gamma_\lambda\gamma_\mu)_{\alpha\beta}Q_\beta \quad , \qquad \text{(B2d)}$$

$$[P_\sigma , Q_\alpha] = 0 \quad , \qquad \text{(B2e)}$$

and
$$[Q_\alpha , Q_\beta] = i/\hbar(\gamma^\sigma C)_{\alpha\beta}P_\sigma \quad . \qquad \text{(B2f)}$$

In our chiral representation we put $Q_{L\frac{1}{2}} = Q^C_1$, $Q_{L-\frac{1}{2}} = Q^C_2$, $Q_{R\frac{1}{2}} = Q^C_3$, $Q_{R-\frac{1}{2}} = Q^C_4$. Here the superscript C indicates the chiral representation of the Dirac matrices as given in equations (A3). These generators satisfy:

$$Q^\dagger_{Ln} = (i\sigma^L_2)_{nn'}Q_{Rn'} \qquad \text{(B2a)}$$

$$Q^\dagger_{Rn} = (i\sigma^R_2)_{nn'}Q_{Ln'} \qquad \text{(B2b)}$$

In terms of these chiral generators and the left and right handed Pauli matrices equations (B1d) and (B1f) become:

$$[M_{\lambda\mu}, Q_{Ln}] = \hbar/i(\sigma_\lambda\sigma_\mu)_{nn'}Q_{Ln'} \quad , \qquad \text{(B3a)}$$

$$[M_{\lambda\mu}, Q_{Rn}] = \hbar/i(\sigma_\lambda\sigma_\mu)_{nn'}Q_{n'} \quad , \qquad \text{(B3b)}$$

$$[Q_{Ln}, Q_{Lr}] = 0 \quad , \qquad \text{(B3c)}$$

$$[Q_{Rn}, Q_{Rr}] = 0 \quad , \qquad \text{(B3d)}$$

and
$$[Q_{Ln}, Q_{Rr}] = i/\hbar(\sigma^R_\sigma\sigma^R_2)_{nr}P_\sigma \quad , \qquad \text{(B3e)}$$

$$= i/\hbar(\sigma^L_\sigma\sigma^L_2)_{rn}P_\sigma \quad .$$

Note that equations (B2) and (B3) are symmetric in the interchange of L and R.

In section II we find it convenient to use the standard operators J_+, J_- and J_3 defined by:

$$J_+ = M_{23} + iM_{31} \quad ,$$

$$J_- = M_{23} - iM_{31} \quad ,$$

and $$J_3 = M_{12} \quad .$$

also in that section we work in terms of the operators Q_{Ln} and $Q^{\dagger}_{Ln}$.

Equations (B3a) and (B3b) then become:

$$[J_+ , Q_{L\frac{1}{2}}] = -\hbar Q_{L-\frac{1}{2}} \quad , \tag{B4a}$$

$$[J_+ , Q_{L-\frac{1}{2}}] = 0 \quad , \tag{B4b}$$

$$[J_- , Q_{L\frac{1}{2}}] = 0 \quad , \tag{B4c}$$

$$[J_- , Q_{L-\frac{1}{2}}] = \hbar Q_{L\frac{1}{2}} \quad , \tag{B4d}$$

$$[J_3 , Q_{L\frac{1}{2}}] = -\hbar/2 Q_{L\frac{1}{2}} \quad , \tag{B4e}$$

and $$[J_+ , Q_{L-\frac{1}{2}}] = \hbar/2 Q_{L-\frac{1}{2}} \quad ; \tag{B4f}$$

and

$$[J_+ , Q^{\dagger}_{L\frac{1}{2}}] = 0 \quad , \tag{B5a}$$

$$[J_+ , Q^{\dagger}_{L-\frac{1}{2}}] = \hbar Q^{\dagger}_{L\frac{1}{2}} \quad , \tag{B5b}$$

$$[J_- , Q^{\dagger}_{L\frac{1}{2}}] = \hbar Q^{\dagger}_{L-\frac{1}{2}} \quad , \tag{B5c}$$

$$[J_- , Q^{\dagger}_{L-\frac{1}{2}}] = 0 \quad , \tag{B5d}$$

$$[J_3 , Q^{\dagger}_{L\frac{1}{2}}] = \hbar/2 Q^{\dagger}_{L\frac{1}{2}} \quad , \tag{B5e}$$

and $$[J_+ , Q^{\dagger}_{L-\frac{1}{2}}] = -\hbar/2 Q^{\dagger}_{L-\frac{1}{2}} \quad . \tag{B5f}$$

REFERENCES

1. A. Salam, J. Strathdee, N.Phys.B80(1974) 499-505.

2. P.Fayet, S.Ferrara, Phys.Repo.32(1977) 249-334.

3. S.Weinberg,Phys.Rev.B133(1964) 1318-1332; Phys.Rev.B134(1964) 882-896; Phys.Rev.181(1969) 1893-1899.

4. E.Sokatchev, N.Phys.B99(1975) 96-108.

5. J.F.Cornwell, *Group theory in Physics, Vols. I & II,* Academic Press, London, 1984.

6. J. Wess, B. Zumino, N.Phys.B70(1974) 39-50; Phys.Lett.49B(1974) 52-54.

7. D.Williams, J.F. Cornwell, *'Unitary Irreducible Representations of Lie Supergroups'*, submitted to J.Math.Phys.