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AN ELEMENTARY APPROACH TO BROWNIAN MOTION ON MANIFOLDS

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§1. Introduction

In this talk I will describe an elementary approach to the study of Brownian motion on manifolds. It arose from reading the paper by Price and Williams [1] on Brownian motion on the unit sphere S^2 in \mathbb{R}^3 . Their results were generalized to a hypersurface in \mathbb{R}^d in [2] and to a submanifold of \mathbb{R}^d of arbitrary co-dimension in [3]. This approach regards all the processes involved as processes on the ambient Euclidean space; it has the advantage that it lends itself to the martingale point of view; it has the disadvantage that all the objects of differential geometry which arise (covariant derivative, second fundamental form, Laplace-Beltrami operator,...) must be defined in an open neighbourhood of the submanifold. The casual reader is warned that there is already an extensive literature on Brownian motion on manifolds in which the differential geometry is treated from the intrinsic point of view; Ellworthy [4] is an excellent guide to this.

We begin by recalling the following equivalent definitions of $BM(\mathbb{R}^d)$, Brownian motion in \mathbb{R}^d :

1. A process B on \mathbb{R}^d with $B_0=0$ is a $BM(\mathbb{R}^d)$ if and only if B_t is gaussian with $E[B_t]=0$ for each t and $E[B_s B_t^T] = (s \wedge t)1$ for each pair s, t . (Here the superscript T denotes 'transpose', and we regard $B_s B_t^T$ as a linear mapping on \mathbb{R}^d ; the identity mapping on \mathbb{R}^d is denoted by 1).

2. A process B on \mathbb{R}^d with $B_0=0$ is a $BM(\mathbb{R}^d)$ if and only if B is a diffusion on \mathbb{R}^d with generator $\frac{1}{2}\Delta$, where Δ is the Laplacian on \mathbb{R}^d .

3. A process B on \mathbb{R}^d with $B_0=0$ is a $BM(\mathbb{R}^d)$ if and only if B is a semimartingale and

(i) $dB_t = dM_t$, where M is a continuous local martingale.

(ii) $d\langle BB^T \rangle_t = 1 dt$. (1.1)

The equivalence of these three definitions is proved in Ikeda and Watanabe [5], for example. The first definition is the most elementary,

but it cannot serve as a model for a definition of a Brownian motion on a manifold because the gaussian property will certainly not survive if the manifold is compact. The second definition is close to Einstein's original treatment and will serve as our model; we require simply that the process remains on the manifold for all t (almost surely), and that it is a diffusion whose generator is $\frac{1}{2}\Delta$, where Δ is now the Laplace-Beltrami operator. The third definition is a version of Lévy's martingale characterization of Brownian motion; it will serve as the model for our main result and it is the keystone of its proof.

§2. Submanifolds of Euclidean Space

We shall consider here submanifolds of \mathbb{R}^d which are level sets of a C^2 -function $f: U \rightarrow \mathbb{R}^r$ defined on an open set U in \mathbb{R}^d . We require that the level set $V=f^{-1}(c)$ be such that the derivative $f'(x)$ is of rank r for all x in V ; then there is an open neighbourhood W of V such that $f'(y)$ has rank r for all y in W . The set W is made up of level sets of f , all having the same dimension. Let T_y be the kernel of $f'(y)$ for each y in W ; then T_y is the tangent subspace at y to the unique level set of f through the point y , and we denote by $P(y)$ the orthogonal projection of \mathbb{R}^d onto T_y . The orthogonal complement T_y^\perp of T_y is the normal subspace at y , and we denote by $P^\perp(y)$ the orthogonal projection of \mathbb{R}^d onto T_y^\perp . We will say that a vector field $v: W \rightarrow \mathbb{R}^d$ is a tangent vector field if $v(y)$ lies in T_y for each y in W ; and we will say that it is a normal vector field if $v(y)$ lies in T_y^\perp for each y in W . Given a pair v, w of tangent vector fields v, w defined on W we decompose the derivative $(v \cdot \text{grad } w)(y)$ of w in the direction of v as

$$(v \cdot \text{grad } w)(y) = (\nabla_v w)(y) + s_y(v, w) \quad (2.1)$$

where

$$(\nabla_v w)(y) = P(y) (v \cdot \text{grad } w)(y), \quad (2.2)$$

and

$$s_y(v, w) = P^\perp(y) (v \cdot \text{grad } w)(y). \quad (2.3)$$

When restricted to V , the tangent vector field $\nabla_v w$ is called the covariant-derivative of w with respect to v , and the normal vector field $s(v, w)$ is called the second fundamental form of the imbedding of V in \mathbb{R}^d . We define another normal vector field j on W by

$$j(y) = \frac{1}{2} \text{trace}_{T_y}(s_y). \quad (2.4)$$

Let $\{n_1, \dots, n_r\}$ be an orthonormal family of normal vector fields on W ; the set $\{n_1(y), \dots, n_r(y)\}$ is an orthonormal basis for T_y^\perp . (We can construct such a family by taking the components $\{f^1, \dots, f^r\}$ of f with respect to an orthonormal basis for \mathbb{R}^r and applying the Gram-Schmidt process to $\{\text{grad } f^1, \dots, \text{grad } f^r\}$). Then

$$\begin{aligned} S_y(v, w) &= \sum_{j=1}^r n_j(y) \{(v \cdot \text{grad } w)(y) \cdot n_j(y)\} \\ &= - \sum_{j=1}^r n_j(y) (v(y) \cdot n_j'(y) w(y)), \end{aligned} \quad (2.5)$$

since $n_j(y) \cdot w(y) = 0$ for $j=1, \dots, r$. Thus

$$j(y) = - \frac{1}{2} \sum_{j=1}^r n_j(y) \text{trace} [P(y) n_j'(y)] \quad (2.6)$$

$$= - \frac{1}{2} \sum_{j=1}^r n_j(y) \{(\text{div } n_j) - \sum_{k \neq j} (n_k(y) \cdot n_j'(y) n_k(y))\}. \quad (2.7)$$

In the case of a hypersurface ($r=1$) the expression (2.7) can be written

$$j(y) = \frac{d-1}{2} H(y) n(y), \quad (2.8)$$

where $H(y)$ is the mean curvature at y of the level surface through y and n is the orienting vector field, while (2.7) yields the computationally useful formula

$$j(y) = - \frac{1}{2} n(y) (\text{div } n)(y). \quad (2.9)$$

The covariant derivative $(\nabla g)(y)$ of a function $g: W \rightarrow \mathbb{R}$ is defined by

$$(\nabla g)(y) = P(y)(\text{grad } g)(y), \quad (2.10)$$

and the Laplace-Beltrami operator Δ by

$$(\Delta g)(y) = \text{trace} ((\nabla^2 g)(y)). \quad (2.11)$$

It follows from (2.10) that

$$\begin{aligned} (\Delta g)(y) &= \text{trace} [P(y)\{\text{grad}(P(y)\text{grad } g(y))\}] = \text{trace} [P(y)g''(y)] \\ &= - \sum_{j=1}^r (\text{grad } g(y) \cdot n_j(y)) \text{trace} [P(y)n_j'(y)]. \end{aligned} \quad (2.12)$$

Rewriting (2.12), using (2.6), we have

$$\frac{1}{2}(\Delta g)(y) = \frac{1}{2} \text{ trace } [P(y)g''(y)] - j(y) \cdot (\text{grad } g)(y), \quad (2.13)$$

an identity which will prove useful in the next section.

§3. Brownian Motion on a Submanifold

Let $V = f^{-1}(c)$ be as described in §2. We claim that a process X in \mathbb{R}^d , with $f(X_0) = c$ and

$$dX_t - j(X_t)dt = P(X_t)dB_t, \quad (3.1)$$

is a BM(V), a Brownian motion on the submanifold $V = f^{-1}(c)$; here B
is a BM(\mathbb{R}^d), a Brownian motion on \mathbb{R}^d .

Now X is a diffusion, since it satisfies an Itô equation; we have to show that its generator is $\frac{1}{2}\Delta$ and that it remains on the surface for $t>0$. Let g be an arbitrary C^2 -function $g:W \rightarrow \mathbb{R}$, and apply Itô's formula to the process $g(X)$:

$$dg(X_t) = (\text{grad } g)(X_t) \cdot dX_t + \frac{1}{2} \text{trace} [g''(X_t) d\langle XX^T \rangle_t]. \quad (3.2)$$

From (3.1) and (1.1) we have

$$d\langle XX^T \rangle_t = P(X_t)dt, \quad (3.3)$$

so that

$$dg(X_t) = dN_t + j(X_t) \cdot (\text{grad } g)(X_t)dt + \frac{1}{2} \text{trace} [g''(X_t)P(X_t)]dt \quad (3.4)$$

where

$$dN_t = (\text{grad } g)(X_t) \cdot P(X_t) dB_t. \quad (3.5)$$

Thus

$$dg(X_t) - \frac{1}{2}(\Delta g)(X_t)dt = dN_t, \quad (3.5)$$

where N_t is a continuous local martingale; we conclude that $\frac{1}{2}\Delta$ is the generator of the diffusion X . It remains to show that X remains on $V = f^{-1}(c)$ for $t > 0$. Let $g = f^j, j=1, \dots, r$; then

$$dN_t = (\text{grad } f^j)(X_t) \cdot P(X_t) dB_t = 0$$

since $(\text{grad } f^j)(y)$ is orthogonal to T_y , and

$$(\nabla f^j)(y) = P(y)(\text{grad } f^j)(y) = 0,$$

for the same reason. It follows from (3.4) that $df^j(X_t) = 0$ for $j=1, \dots, r$. Thus X stays on $V = f^{-1}(c)$ for $t > 0$ since it starts there.

Remark: The equation (3.1) for Brownian motion on a submanifold of Euclidean space was given by Baxendale [6].

§4. Martingale Characterization

The description of Brownian motion on $V = f^{-1}(c)$ given in §3 suggests the following

Martingale Characterization of BM(V):

A process X on \mathbb{R}^d with $f(X_0) = c$ is a BM(V) if and only if X is a semimartingale such that

(1) $dX_t - j(X_t)dt = dM_t$, where M is a continuous local martingale.

(2) $d\langle XX^T \rangle_t = P(X_t)dt$.

We have to show that, given a semimartingale on \mathbb{R}^d satisfying (1) and (2), there exists B , a BM(\mathbb{R}^d), such that

$$dM_t = P(X_t)dB_t. \quad (4.1)$$

Let \tilde{B} be a BM(\mathbb{R}^d) which is independent of X , so that

$$d\langle \tilde{B}\tilde{B}^T \rangle_t = I dt, \quad d\langle X\tilde{B}^T \rangle = 0, \quad (4.2)$$

and let \tilde{B} be a process on \mathbb{R}^d such that $B_0=0$ and

$$d\tilde{B}_t = P(X_t)dX_t + P^\perp(X_t)d\tilde{B}_t; \quad (4.3)$$

then by (2) and (4.2) we have

$$d\langle \tilde{B}\tilde{B}^T \rangle = P(X_t)dt + P^\perp(X_t)dt = I dt. \quad (4.4)$$

It follows, by the martingale characterization of $BM(\mathbb{R}^d)$, that \tilde{B} is a $BM(\mathbb{R}^d)$ and, by (1), that

$$P(X_t)dM_t = P(X_t)dB_t. \quad (4.5)$$

It remains to show that $P(X_t)dM_t = dM_t$. Consider the process N on \mathbb{R}^d such that $N_0=0$ and

$$dN_t = P^\perp(X_t)dM_t. \quad (4.6)$$

Then

$$d\langle NN^T \rangle_t = P^\perp(X_t)P(X_t)P^\perp(X_t)dt = 0, \quad (4.7)$$

so that NN^T is also a continuous local martingale; but NN^T is non-negative so that NN^T is constant almost surely, and so $dN_t=0$ and

$$dM_t = P(X_t)dM_t = P(X_t)dB_t. \quad (4.8)$$

§5. Examples

(1) Hypersurfaces in \mathbb{R}^d

In this case, $r=1$ and

$$j(x) = \frac{d-1}{2}H(x)n(x), \quad (5.1)$$

where $H(x)$ is the mean curvature of V at x , and n is the orienting normal vector field. Then a $BM(V)$ is a martingale in the ambient Euclidean space if and only if the mean curvature of V vanishes identically. (Compare [7]).

It follows from (3.1) that, if X is such that

$$dX_t - \frac{(d-1)}{2}H(X_t)n(X_t)dt = P(X_t)dB_t, \quad (5.2)$$

then X is a $BM(V)$. It follows from the martingale characterization that an alternative equation for $BM(V)$ is

$$dX_t - \frac{(d-1)}{2}H(X_t)n(X_t)dt = d\tilde{B}_t n(X_t), \quad (5.3)$$

where \tilde{B} is a $BM(\mathfrak{so}(d))$, a Brownian motion in the Lie algebra of the

orthogonal group $SO(d)$, since $d\langle XX^T \rangle_t = P(X_t)dt$; see [2].

(2) The unit sphere S^2 in \mathbb{R}^3

In the special case of S^2 , the unit sphere in \mathbb{R}^3 , we take $n(x)=x$, the outward normal at x ; then the principal curvatures are both equal to -1 , so that $j(x)=-x$. The projection $P(x)$ onto the tangent space at x is given by $P(x) = (1-xx^T)$. Then (5.2) yields the equation of Stroock[7]:

$$dX_t + X_t dt = (1 - X_t X_t^T) dB_t. \quad (5.4)$$

On the other hand, (5.3) yields the equation of Price and Williams [1]:

$$dX_t + X_t dt = X_t \times dB_t. \quad (5.5)$$

(3) Curves in \mathbb{R}^d

Let $s \rightarrow x(s)$ be a C^2 -curve in \mathbb{R}^d , parametrized by arc length; then the tangent vector $t(s)$ at $x(s)$ is given by

$$t(s) = \frac{dx}{ds}(s) \quad (5.6)$$

and

$$\frac{dt}{ds}(s) = k(s)n(s) \quad (5.7)$$

where $n(s)$ is the principal normal at $x(s)$ and $k(s)$ is the curvature. Then

$$j(x(s)) = \frac{1}{2} k(s)n(s), \quad (5.8)$$

and

$$P(x(s)) = t(s)t(s)^T. \quad (5.9)$$

Now let b be a $BM(\mathbb{R}^1)$ and put $X_t = x(b_t)$. Then X is a process in \mathbb{R}^d beginning at $x(0)$ and

$$dX_t = \frac{dx}{ds}(b_t)db_t + \frac{1}{2} \frac{d^2x}{ds^2}(b_t)dt = t(b_t)db_t + \frac{1}{2} k(b_t)n(b_t)dt, \quad (5.10)$$

so that

$$dX_t - j(X_t)dt = t(b_t)db_t. \quad (5.11)$$

It follows from (5.11) that

7.

$$d\langle XX^T \rangle_t = t(b_t)t(b_t)^T dt; \quad (5.12)$$

using (5.8) we have

$$d\langle XX^T \rangle_t = P(X_t)dt.$$

By the martingale characterization, it follows that X is a Brownian motion on the curve $s \rightarrow x(s)$.

§6. Martingale Representation

Let X be a Brownian motion on $V = f^{-1}(c)$ starting at x , and let Y be defined by $Y_0=0$ and

$$dY_t = P(X_t)dX_t, \quad (6.1)$$

so that dY_t is the tangential component of dX_t . Let \tilde{X} be another Brownian motion on $V = f^{-1}(c)$ starting at x , and let \tilde{Y} be defined by $Y_0=0$ and

$$d\tilde{Y}_t = P(\tilde{X}_t)d\tilde{X}_t. \quad (6.2)$$

Suppose that \tilde{X} is adapted to the filtration of X ; then we have the following

Martingale Representation: The processes Y and \tilde{Y} are related by the Itô equation

$$d\tilde{Y}_t = C_t dY_t \quad (6.3)$$

where

(1) for each t , C_t is an orthogonal transformation such that

$$C_t n(X_t) = n(\tilde{X}_t) \quad (6.4)$$

for each unit normal vector field n on V .

(2) the process C is X -predictable.

Let $\{n_1, \dots, n_r\}$ be an orthonormal set of normal vector fields on V ; let $\{b^1, \dots, b^r\}$ be a set of independent $BM(\mathbb{R}^1)$ -processes independent of both X and \tilde{X} so that

$$d\langle X^i b^j \rangle = d\langle \tilde{X}^i b^j \rangle = 0, \quad i, j=1, \dots, r, \quad (6.5)$$

and

$$d\langle b^i b^j \rangle = \delta_{ij} dt. \quad (6.6)$$

Then, by the argument in §4, the processes B and \tilde{B} such that $B_0 = \tilde{B}_0 = 0$ and

$$dB_t = dY_t + \sum_{j=1}^r n_j(X_t) db^j, \quad d\tilde{B}_t = d\tilde{Y}_t + \sum_{j=1}^r n_j(\tilde{X}_t) db^j \quad (6.7)$$

are both $BM(\mathbb{R}^d)$ and X and \tilde{X} satisfy

$$dX_t - j(X_t)dt = P(X_t)dB_t, \quad d\tilde{X}_t - j(\tilde{X}_t)dt = P(\tilde{X}_t)d\tilde{B}_t. \quad (6.8)$$

Moreover, \tilde{B} is B -predictable so that, by the martingale representation theorem for $BM(\mathbb{R}^d)$, there exists a B -predictable process C of orthogonal transformations on \mathbb{R}^d such that

$$d\tilde{B}_t = C_t dB_t. \quad (6.9)$$

Hence, from (6.7), we have

$$C_t dY_t + \sum_{j=1}^r C_t n_j(X_t) db^j = d\tilde{Y}_t + \sum_{j=1}^r n_j(\tilde{X}_t) db^j; \quad (6.10)$$

Forming the bracket process of both sides with the process b^k , using (6.5) and (6.6), we have

$$C_t n_k(X_t) dt = n_k(\tilde{X}_t) dt, \quad (6.11)$$

establishing (6.4), and (6.3) follows by subtraction. It follows from (6.4) that C can be chosen to be X -predictable.

Special Cases:

- (1) For a hypersurface ($r=1$); taking n to be the orienting vector field, the map $x \rightarrow n(x)$ is the Gauss map.
- (2) Specializing to S^2 , the unit sphere in \mathbb{R}^3 , we have $n(x)=x$ and we recover the result of Price and Williams [1]:

Let X and \tilde{X} be $BM(S^2)$ - processes starting at x ; suppose that \tilde{X} is adapted to the filtration of X . Then the tangential increments dY and $d\tilde{Y}$ are related by the Itô equation

$$d\tilde{Y}_t = C_t dY_t \quad (6.12)$$

where (1) for each t , C_t is an orthogonal transformation such that

$$C_t X_t = \tilde{X}_t, \quad (6.13)$$

(2) the process C is X -predictable.

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