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The "Symmetry-Conservation Law" duality in G.R. via a new approach to projective differential geometry.

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Introduction The main idea is that use of tangent bundle geometry allows a vector field approach to all geodesic conservation laws not just the linear ones associated with isometries. The details are developed in Prince and Crampin [1], a summary of which is soon to appear (Prince [2]). I will restrict myself to a couple of interesting results, namely that the Noether-Hamilton-Cartan theorem shows that homothetic actions act on $\mathbb{R}\times\mathcal{M}$ and that they correspond to a universal conservation law (not just a null one) and also that Killing tensors correspond to a one parameter group action on $E=\mathbb{R}\times T\mathcal{M}$ and that a projective version of a conformal Killing tensor corresponds to a higher order analogue of a homothetic motion.

Notation ($\mathcal{M}, g$) is the spacetime with local co-ordinates (x^a), $\mathbb{R}\times\mathcal{M}$ co-ordinates (s, x^a), is needed for affinely parametrised geodesics. $E=\mathbb{R}\times T\mathcal{M}$ is the evolution space, co-ordinates (s, x^a, u^a).

Symmetry on E Only on evolution space is there a vector field (one-parameter group) representing the geodesics; locally this is

$$\Gamma = \frac{\partial}{\partial s} + u^a\frac{\partial}{\partial x^a} - \Gamma^a_{bc}u^b u^c\frac{\partial}{\partial u^a}.$$

In a Lagrangian description Γ is the characteristic vector field of the exterior derivative of the Cartan form θ (the fundamental 1-form of Hamiltonian mechanics $pdq - Hdt$ pulled back by the Legendre transformation). In terms of the dual bases

$$\left\{\Gamma, H_a := \frac{\partial}{\partial x^a} - u^b\Gamma^c_{ba}\frac{\partial}{\partial u^c}, V_a := \frac{\partial}{\partial u^a}\right\} \quad \text{and}$$

$$\left\{ds, \omega^a := dx^a - u^a ds, \theta^a := du^a - u^c\Gamma^a_{bc}dx^b\right\}$$

of TE and T^*E (H_a, V_a are horizontal and vertical fields respectively),

and

$$\theta = L ds + L_{,\dot a}\omega^a$$
$$d\theta = g_{ab}\theta^a\wedge\omega^b$$

where $L = g_{ab}u^a u^b$ and $L_{,\dot a} := \dfrac{\partial L}{\partial u^a}$.

The *projective actions* on E are generated by the dynamical symmetries of Γ, that is vector fields Z with

$$\mathcal{L}_Z\Gamma = \lambda\Gamma \quad , \lambda \in \mathcal{F}(E).$$

If Z generates a flow on E which has been lifted from $\mathbb{R}\times\mathcal{M}$ then $Z = X^{(1)}$ where $X^{(1)}$ is the prolongation of the generator X of the flow on $\mathbb{R}\times\mathcal{M}$. In this case X (a *Lie symmetry*) generates a classical projective action on along with the "induced s-transformation".

There a number of ways of associating geodesic first integrals with projective actions on E, however the Noether-Hamilton-Cartan theorem gives a clean cut result: *if locally a function $f \in \mathcal{F}(E)$ can be found such that $\mathcal{L}_Y \theta = df$ for a given vector field $Y \in \mathcal{X}(E)$ then $F = f - \langle Y, \theta \rangle$ is a local constant of the motion. Conversely if F is a (local) constant of the motion then there exists a unique (mod Γ) Y such that*

$$dF = Y \lrcorner d\theta \iff \mathcal{L}_Y \theta = d(F + \langle Y, \theta \rangle).$$

Such vector fields are called *Cartan symmetries* and they are all projective actions on E. When $Y := X^{(1)}$, X is just a *Noether symmetry*, that is, one given by the classical Noether theorem.

The classical isometry (Killing vector) is such a Noether symmetry: $X = \xi^a \frac{\partial}{\partial x^a}$, $F = \xi_a u^a$. So are the homothetic actions

$$X = s\frac{\partial}{\partial s} + \xi^a \frac{\partial}{\partial x^a}, \quad \xi^a \in \mathcal{F}(\mathcal{M}), \quad F = sL - \xi_a u^a.$$

Notice that homothetic actions give universal geodesics first integrals not just null ones. (The complete list of Lie and Noether symmetries appears in [1].)

The Cartan symmetry corresponding to the Killing tensor first integral $K_{ab}u^a u^b$ is

$$Y = -2K^a{}_b u^b \frac{\partial}{\partial x^a} + \Gamma(-2K^a{}_b u^b)\frac{\partial}{\partial u^a},$$

$$\mathcal{L}_Y \theta = d(-K_{ab}u^a u^b).$$

The interesting feature of this *Killing motion* on E is that

$$(\mathcal{L}_Y g)(\Gamma, \Gamma) = 0$$

(c.f. $\mathcal{L}_X g = 0$ for Killing vectors).

Conformal Killing tensors don't fit into the scheme because they don't generate universal first integrals, however, an attempt to find a quadratic analogue to the homothetic constant of the motion yields (essentially uniquely)

$$F = s\zeta_a u^a L - H_{ab}u^a u^b$$

where $W = \zeta^a \frac{\partial}{\partial x^a}$ is a Killing vector and H is symmetric and tracefree with

$$H_{(ab;c)} = g_{(ab}\zeta_{c)}$$

The corresponding Cartan symmetry has local co-ordinate components

$$\sigma = s\zeta_a u^a,\ \xi^a = 2H^a_{\ b}u^b - s\zeta^a L,\ \eta^a = \Gamma(\xi^a) - u^a\Gamma(\sigma)$$

and $\quad \mathcal{L}_Y \Theta = -dF.$

This is called a *homothetic action on* E because

$$(\mathcal{L}_Y g)(\Gamma,\Gamma) = \pi^* g(\Gamma, W^{(1)})\pi^* g(\Gamma,\Gamma)$$

or

$$Y(L) = -u^a \zeta_a L$$

by analogy with $\mathcal{L}_X g = a g,\ a \in \mathbb{R}$ for the classical homothetic motion. Notice that the "homothetic constant" $\pi^* g(\Gamma, W^{(1)})$ is now a constant of the motion.

Remarks There are lots of results, new and old, which are readily accessible using this sort of tangent bundle geometry. We have had some success relating projective actions on E with global Jacobi fields (solutions of the equation of geodesic deviation), the Raychaudhuri equation [3] and in defining conformal motions on E to correspond to conformal Killing tensors. I hope to explore some of this structure for particular metrics in the near future.

References

[1] Prince, Geoff and Crampin, Mike (1983) Projective differential geometry and geodesic conservation laws in general relativity. DIAS / Open University Preprint.

[2] Prince, Geoff (1983) Homothetic Killing tensors. Phys. Lett A (to appear).

[3] Crampin, Mike and Prince, Geoff (1983) Open University / DIAS Preprint.

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