DYNAMICAL GROUPS AND COEXISTENCE PHENOMENA

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ABSTRACT

This note describes an application of dynamical Lie groups to many body systems exhibiting phase transitions. The specific model exemplified is that of a three-phase many fermion system for which the appropriate group is SO(6).

1. Foreword

The following talk describes what I think is a fairly unusual application of Lie algebras, to the solution of problems involving collective phenomena. I was first introduced to Lie algebras, and more specifically spectrum-generating Lie algebras in the context of particle physics, while visiting Professor Yuval Ne'eman at the Department of Physics of Tel Aviv University in 1968-69. It gives me great pleasure to acknowledge the influence that he exercised on my subsequent work, and I am very happy to be able to contribute to the Proceedings of this year's Group Theory Conference, the year in which Professor Ne'eman received the Wigner Medal for his outstanding contributions in the field.

2. Dynamical Groups

The association of symmetry groups with phase transitions is well-founded(1). The system in question in its disordered state, above a critical temperature T_C, is described by a hamiltonian H having symmetry group G_S. In the ordered state, below T_C, the system is conventionally described by a reduced hamiltonian H_{red} which is invariant under a smaller symmetry group $G_B \subset G_S$.

Dynamical groups - non-symmetry groups of the system, arise in the following way: the reduced hamiltonian, usually a mean-field approximation, is exactly solvable (diagonalizable), and is a representation of an element of a Lie algebra, the so-called Spectrum Generating Algebra (SGA). A working definition of the dynamical group would be the Lie group of this SGA. The best-known example is the so(4,2) SGA of the hydrogen atom(2); the name of the algebra derives from its property of generating the spectrum of the system. This valuable property is shared by the SGA's of solvable many-body models; such SGA's usually have the form of a direct sum of Lie algebras indexed by momentum, $g = \bigoplus_k g_k$, where each g_k is isomorphic to a fixed

Lie algebra (which we shall occasionally loosely refer to as the SGA of the system). An example is the SGA for an exactly solvable model(3) of superfluid Helium Four; here $g \sim \bigoplus_k su(1,1)_k$.

3. Quadratic Hamiltonians

A general example of such an exactly-solvable system will be given by a reduced hamiltonian H_{red} which is expressed as a direct sum of (essentially one-particle) commuting hamiltonians indexed by momentum k, where each H_k is quadratic(4) in some set of creation and annihilation operators given by a column d-vector A_k. Thus

$$H_{red} = \sum_k H_k$$

$$H_k = A_k^+ M_k A_k$$

where M_k is a hermitian $d \times d$ matrix over $\mathbb{C}$. This is sufficient to ensure that H_{red} is a representative of an element x_{red} of a Lie algebra $g \sim \bigoplus_k g_k$. If we assume for definiteness that we have a fermion system

$$\{A_{ik}, A_{jk}^+\} = \delta_{ij}$$

then each g_k is isomorphic to a subalgebra of u(d).

4. Bogoliubov Transformation

In general, we may choose a Cartan basis for a rank-ℓ, n-dimensional semi-simple complex Lie algebra g

$$\{h_1, h_2, \ldots, h_\ell;\ e_1, e_2, \ldots, e_{n-\ell}\} \tag{1}$$

where
$$[h_i, h_j] = 0, \quad [h_i, e_\alpha] = \alpha_i e_\alpha \tag{2}$$
$$[e_\alpha, e_\beta] = N_{\alpha+\beta} e_{\alpha+\beta} \quad (\text{if } \alpha + \beta \neq 0),$$
$$[e_\alpha, e_{-\alpha}] = \alpha^i e_\alpha .$$

The diagonalizable H_{red} is a representative of a semi-simple element $x_{red} \in g$; as a consequence, there exists an automorphism $\phi : g \to g$ which sends x_{red} into the commuting Cartan Subalgebra (CSA) generated by $\{h_i\}$[5]. This automorphism will be implemented in the d-dimensional representation by a rotation R, and in Fock space by a unitary transformation U, as in diagram 1.

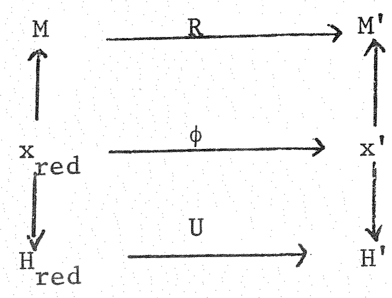

Diagram 1

Thus the matrix $M' = RMR^+$ is diagonal; and if we define new operators $\mathcal{A}_k$ by

$$\mathcal{A}_k = U^+ A_k U$$

then, expressed in terms of the new operators $\mathcal{A}_k$, the reduced hamiltonian H_{red} is diagonal, $H_k = \mathcal{A}_k^+ M'_k \mathcal{A}_k$. The transformation $A_k \longmapsto \mathcal{A}_k = U^+ A_1 U = RA_k$ is known as the Bogoliubov transformation[6] from its first use in solving the Helium Four problem.

5. Order Parameters

There is a certain arbitrariness in thechoice of the automorphism ϕ. In the case where the symmetry group G_S of the original hamiltonian H (in the disordered state, about T_C) is abelian, we may choose the corresponding commutative algebra g_s to be a sub-algebra of the CSA of g. In this case, the element $x_{sym} \in g$ defined by $x_{sym} = \phi(x_{red}) = \sum_{i=1}^{\ell} \mu_i h_i$ 'recovers' the original symmetry of H. If we take its Fock space representative H_{sym} as a 'good' (in the group theoretical sense) description of the orignal system (above T_C), we see that the elements e_α of the Cartan basis (1) behave as order parameters for the system[7]. This follows from the commutation relations (2)

$$[x_{sym}, e_\alpha] = [\sum_i \mu_i h_i, e_\alpha]$$

$$= (\sum_i \mu_i \alpha_i) e_\alpha .$$

Taking the expectation value of this commutator in eigenstates of H_{sym}, we have

$$<[H_{sym}, \hat{e}_\alpha]>_{sym} = \sum_i \mu_i \alpha_i <\hat{e}_\alpha>_{sym}$$

$$= 0.$$

If $\sum_i \mu_i \alpha_i \neq 0$, this shows that the Hilbert space representative $\hat{e}_\alpha$ of e_α vanishes in eigenstates of H_{sym} - representing the system above T_C - and is thus a good candidate for an order parameter. In the case of the BCS model of superconductivity, this leads to the well-known local complex order parameter; a similar result is obtained for a u(2) model of charge-density waves[8] (since the considerations above for semi-simple algebras also hold good for the reductive u(2)).

6. so(6) Coexistence Model

We now illustrate the preceding ideas by a many-fermion model which exhibits coexisting phases. Our starting point is thus the anti-commutation relations

$$\{a_{k\sigma}, a^+_{k'\sigma'}\} = \delta_{kk'} \delta_{\sigma\sigma'}$$

for fermions of wave vector k,k' and spin σ,σ'. We define a four-component operator A_i by

$$(A_1, A_2, A_3, A_4)_{(k)} = (a_{k\uparrow}, a^+_{-k\downarrow}, a_{\bar{k}\uparrow}, a^+_{-\bar{k}\downarrow})$$

where $\bar{k} = k + Q$. Here Q is a characteristic wave vector for the physical problem; in thecase we shall discuss, that of charge-density waves (CDW) and anti-ferromagnetic order (AF), $Q = 2k_F$, where k_F is the Fermi surface wave vector. Defining $X_{ij} = A^+_i A_j$ (suppressing the implict k-dependence) we see that

$$[X_{ij}, X_{i'j'}] = \delta_{ji'} X_{ij'} - \delta_{ij'} X_{i'j} .$$

Thus the X_{ij} generate $g\ell(4,R)$; or , as we have only hermitian combinations of operators, $u(4)$. We take as our mean field hamiltonian $H_{red} = \sum_k H_k$, where $H_k = A^+MA$ for some (hermitian) matrix M. Identification of the kinetic energy term as $\epsilon(k) = \epsilon(-k)$ leads to a <u>traceless</u> M; we are therefore dealing with the SGA $su(4) \sim so(6)$. Typical terms occuring in the model are

$x_{12} = a^+_{k\uparrow}a^+_{-k\downarrow}$; superconductor (SC) pairing,

$x_{13} = a^+_{k\uparrow}a_{\bar{k}\uparrow}$; CDW term,

$x_{14} = a^+_{k\uparrow}a^+_{-\bar{k}\downarrow}$; AF anomalous pairing(9).

A Cartan basis for this 15-dimensional rank-3 Lie algebra has the form

$$\{h_1,h_2,h_3;\ e_1,e_2,\ldots,e_{12}\}.$$

We may identify each of the h_i with an operator conserved above T_C; thus

$h_1 \sim N$ Number operator, (sum of k and $\bar{k}$ numbers)

$h_2 \sim P$ Linear (one-dimensional) momentum

$h_3 \sim A$ Anomalous number, (difference of k and $\bar{k}$ numbers).

Physically, it is clear we can recover subphases by considering those operators commuting with h_1, h_2 and h_3 in turn. Thus, the SC terms will certainly not commute with $h_1 \sim N$. Mathematically, this process of taking the centralizer $C(h_i)$ of each h_i in turn leads to a sub-algebra, which we may identify with the SGA of a subphase(10). Each centralizer is given by

$$C(h_i) \sim s(u(2) \oplus u(2)) \sim u(1) \oplus so(4).$$

We therefore obtain the following model, diagram 2: this model, in the absence of magnetic terms, has been previously treated in some detail(11).

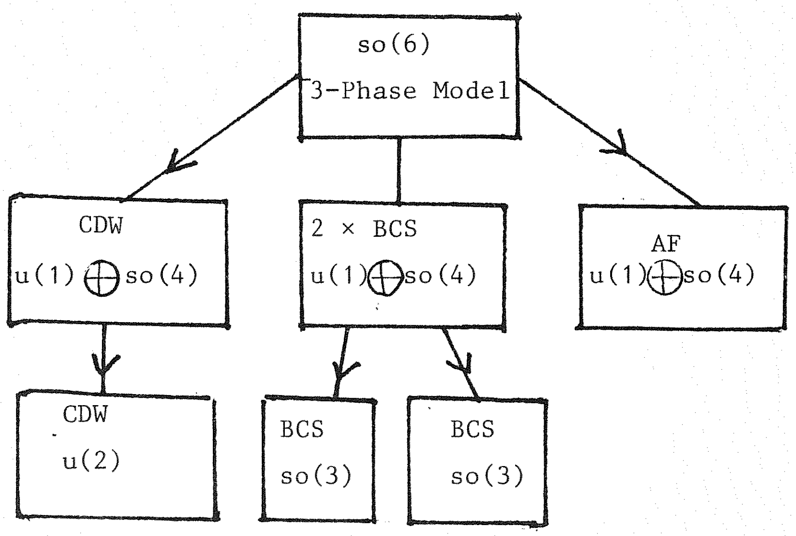

Diagram 2

According to our previous remarks, the order parameters are e_α; and they will determine the presence or absence of phases. Following the root-vector diagram analysis of Van der Waerden (1933) as quoted by Wybourne[12], we see that the root-vectors for so(6) are given by the 12 combinations $\pm\underline{u}_i \pm \underline{u}_j$, where $\underline{u}_1 = (1,0,0)$, $\underline{u}_2 = (0,1,0)$ and $\underline{u}_3 = (0,0,1)$. (As in the accompanying diagram 3.)

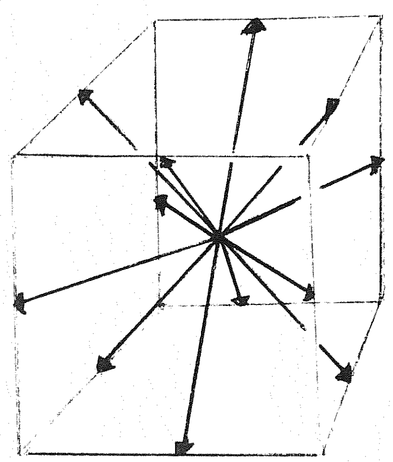

Diagram 3

One immediate conclusion that we may draw is that no root-vector can have non-vanishing components along all 3 axes. For each α, the corresponding local order parameter $\Phi_\alpha(x)$ defined by

$$\Phi_\alpha(x) = \sum_k e_\alpha(k)e^{ikx}$$

must commute with at least one of the h_i(N,P or A) and hence cannot serve as a simultaneous order parameter for more than two of the subphases. This implies the impossibility of the simultaneous coexistence of all three phases.

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