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Finite Size Scaling and the Renormalization Group

Denjoe O'Connor

Dublin Institute for Advanced Studies
10 Burlington Rd.
Dublin 4
Ireland.

C. R. Stephens

Department of Physics,
Imperial College,
London SW7 2BZ, U. K.

ABSTRACT: In this letter we investigate finite size scaling using renormalization group arguments. By employing an L dependent subtraction scheme, we obtain an alternative formulation of finite-size scaling, wherein the scaling variable is $\frac{L}{\xi_L}$, ξ_L being the correlation length in the finite system. This new formulation reduces to the standard one when the only infrared fixed point is the bulk one and/or in the limit $\frac{L}{\xi_L} \to \infty$.

Since all "experimental"systems consist of a finite number of atoms; where by "experimental"we mean both laboratory experiments and computer simulations of lattice systems, all thermodynamic quantities are analytic, however, in the limit of an infinite number of lattice points these quantities can become singular and do so at a critical point. It is of fundamental importance to understand how such singularities arise in this limit. Finite size scaling [1-2] has become an important tool in the investigation of this limit, however a full understanding of finite size scaling is still lacking, in addition agreement with experiment [3] has been less than adequate.

Our current understanding of the singularities arising in the bulk theory relies on the renormalization group (RG) and the existence of fixed points of the ensuing transformations. Finite size scaling (FSS) was investigated using RG techniques by Suzuki [4] and later from a field theoretic point of view by Brézin [5]. Brézin's treatment deals with two types of geometry: A) a finite system characterized by some length scale L in all directions, and B) a system infinite in one dimension and of finite cross section in the others. Both these geometries for finite L forbid the possibility of singular thermodynamic functions. Suzuki treats these two geometries and a third, C) where one has two infinite and one finite direction. Both concluded that the relevant scaling variable is $\frac{L}{\xi_\infty}$ where ξ_∞ is the bulk correlation length. This is in accord with Fisher's original ansatz [1]. One might well wonder though why it is ξ_∞ and not ξ_L, the correlation length in the finite system, that sets the scale.

In this letter we develop an alternative formulation of finite size scaling based on the solutions of an L dependent RG equation. This new formulation seems different to the standard one in the following respects: firstly the relevant scaling variable is $\frac{L}{\xi_L}$ not $\frac{L}{\xi_\infty}$; in addition to the variable $\frac{L}{\xi_L}$ there is an apparent dependence on the coupling constant u in the crossover region, i.e. where $\frac{L}{\xi_L} \sim 1$; lastly for a thermodynamic observable P the ratio $\frac{P_L}{P_\infty}$ is not obviously expressible in scaling form. The advantages of the new formulation from an RG point of view are, that it is capable of treating systems with a fixed point in the finite system and that thermodynamic observables can be computed perturbatively throughout the crossover.

We give a uniform treatment for $d < d_c$, d_c being the upper critical dimension, valid, not only in the previously treated cases, but also when there is a transition in the finite size setting, i.e. for finite L. This will be our main area of concern. Our treatment accommodates the crossover of the critical exponents from those of the bulk critical point to the reduced critical point. We find scaling to be valid when $\frac{L}{\xi_L} \to \infty$ ($\xi_L \to \infty$) with the scaling variable Lt^{ν} where $t = T - T_c(L)$ and ν is the bulk correlation exponent, and when $\frac{L}{\xi_L} \to 0$ with the scaling variable $Lt^{\nu'}$ where ν' is the lower dimensional exponent, i.e. the critical exponent of the theory defined by the lowest mode of the theory in finite L. The former corresponds to the standard theory of FSS.

The RG equation arises as an expression of the fact that the bare theory is independent of the renormalization point at which we have chosen to represent the physical amplitudes of the theory. If Z_ϕ is the wave function renormalization constant, the renormalized and bare N point functions are related by $\Gamma^N_R = Z_\phi^{N/2}\Gamma^N_B$. For a renormalizable theory we will typically have in addition to wave function renormalization, mass renormalization and coupling constant renormalization. The RG equation we get then depends on how we choose our counterterms. In standard minimal subtraction, used in conjunction with dimensional regularization and an ϵ expansion, the counterterms take on a particularly simple form, in that they are independent of the mass. We know that the ultraviolet divergences of the theory in a box with periodic boundary conditions are the same as the bulk theory, since these arise from the short distance fluctuations of the theory, consequently one can use the same set of counterterms for the finite system as for the bulk system. This is the basis of Breźin's

proof [5] of finite size scaling for geometries A) and B) in the presence of an infrared fixed point, as $L \to \infty$ and below the upper critical dimension. Elegant as Brézin's proof is however, it gives no insight into the case where there is a critical point in the finite size setting. The difference there being that the correlation length can now diverge for fixed L, something which is prohibited in the geometries he discusses. In this latter case one would expect that the N point functions would go over to those of the lower dimensional system as $\frac{L}{\xi_L} \to 0$. Now, although one can eliminate all the ultraviolet divergences using the bulk counterterms, one is not restricted to minimal subtraction, or choosing ones counterterms to be just those of the bulk system. One can in fact absorb any amount one wishes of the finite contributions to the diagrams into the counterterms. This will change the RG equation one obtains, and a judicious choice may allow one to extract more information from the theory.

In fact as shown in [6] it is essential to use an L dependent subtraction scheme if one wishes to recover perturbatively the dimensionally reduced system, which arises in the limit $\frac{L}{\xi_L} \to 0$, without encountering new divergences. One way of implementing such a scheme is to choose a non minimal subtraction that includes all terms that diverge as $\kappa L \to 0$ or $\kappa L \to \infty$, the two limits we wish to consider. This gives us counterterms that are L dependent but still mass independent, thereby preserving most of the advantages of the minimal prescription. In this case we get β-functions and anomalous scaling dimensions that are L dependent.

The Lagrangian we will consider, to make our conventions explicit is,

$$L = \frac{1}{2}Z_\phi[(\nabla\phi)^2 + m_B^2\phi^2] + \frac{u_B}{4!}\kappa^\epsilon Z_\phi^2\phi^4 + \frac{1}{2}t_B Z_\phi\phi^2 \qquad (1)$$

The Lagrangian is Fourier expanded in the finite directions and treated as an infinite sum of interacting fields, associated with the Fourier modes. The upper critical dimension, about which one performs an ϵ expansion, occurs when there are four noncompact dimensions. The counterterms in our non-minimal prescription are chosen to have explicit dependence on L by requiring that all quantities divergent as $\frac{L}{\xi_L} \to 0$ are included as well as the usual pole $\epsilon \to 0$. The mass counterterm δm^2 is chosen so that $m^2 = m_B^2 + \delta m^2$ is zero, i.e. we expand about the critical point at finite L, when such exists. The subtractions are dependent now on u and κL, in contrast to the usual situation where they only depend on u. This means that the RG equation takes the form.

$$\left(\kappa\frac{\partial}{\partial\kappa} + \beta(u,\kappa L)\frac{\partial}{\partial u} + \gamma_{\phi^2}(u,\kappa L)t\frac{\partial}{\partial t} - \frac{N}{2}\gamma_\phi(u,\kappa L)\right)\Gamma^{(N)}(k_i, L, u, t, \kappa) = 0 \qquad (2)$$

where $\gamma_{\phi^2} = -\kappa\frac{\partial}{\partial\kappa}\ln Z_{\phi^2}$, $\gamma_\phi = \kappa\frac{\partial}{\partial\kappa}\ln Z_\phi$ and $\beta(u,\kappa L) = \kappa\frac{\partial}{\partial\kappa}u$ The Wilson functions are such that as $\kappa L \to \infty$ they become the relevant functions of the bulk theory, whereas when $\kappa L \to 0$ they become those of the lower dimensional theory, (see [6] for more details). Note, we will only get scaling as $\frac{L}{\xi_L} \to \infty$ if the $L = \infty$ theory is renormalizable. If this is not the case we will recover Brézin's result that finite size scaling is not valid as $L \to \infty$ for $d > d_c$.

Equation (2) can be solved in the standard manner by the method of characteristics. Defining $t(\kappa_0) = t$, $\frac{\kappa}{\kappa_0} = \rho$ and solving the characteristic equation for $t(\kappa)$, we obtain

$$t(\rho) = t\exp\left(\int_1^\rho \gamma_{\phi^2}(u(x, x\kappa_0 L), x\kappa_0 L)\frac{dx}{x}\right)$$

We note that $t = T - T_c(L)$ rather than $T - T_c(\infty)$, this choice is essential if one is to get a sensible dimensionally reduced limit. Similarly solving the beta function equation gives us $u(\rho)$. For simplicity we suppress the L dependence of γ_ϕ, γ_{ϕ^2} and u, and drop the subscript on κ_0. Using the solution to equation (2)

$$\Gamma^N(k_i, L, u, t, \kappa) = \exp\left(\frac{N}{2}\int_\rho^1 \gamma_\phi(x)\frac{dx}{x}\right)\Gamma^N(k_i, L, u(\rho), t(\rho), \rho\kappa) \qquad (3)$$

where $u(1) = u$, and ρ is arbitrary, i.e. the right hand side of equation (3) is independent of ρ, this is simply the content of the RG equation. Using dimensional

analysis one can extract the dependence on $\rho\kappa$ as the overall dimensionful quantity and re-express equation (3) as

$$\Gamma^N(k_i, L, u, t, \kappa) = (\kappa\rho)^{d - N\frac{(d-2)}{2}}\exp\left(\frac{N}{2}\int_\rho^1 \frac{dx}{x}\gamma_\phi(x)\right)\Gamma^N\left(\frac{k_i}{\rho\kappa}, \rho\kappa L, \frac{t(\rho)}{\rho^2\kappa^2}, u(\rho), 1\right) \qquad (4)$$

We now eliminate ρ by choosing it such that

$$\frac{t(\rho)}{\rho^2\kappa^2} = 1 \qquad (5)$$

This gives us an expression for $\rho = \rho(\frac{t}{\kappa^2}, \kappa L)$. Γ^N now depends on $\rho\kappa L$, the running coupling constant and $\frac{k_i}{\rho\kappa}$ if we choose not to set the external momenta to zero.

The most convenient object to work with is a RG invariant. We observe that Γ^N deviates from an RG invariant by the power of ρ and the exponential. If we can find some way of eliminating this prefactor then we will have an invariant. This can be done by forming the ratio

$$R^N = \left(\frac{\Gamma^N}{\Gamma}\right)/\left(\frac{\Gamma^2}{\Gamma}\right)^{\frac{N}{2}}$$

Since R^N is an RG invariant,

$$R^N\left(\frac{k_i}{\kappa}, \kappa L, \frac{t}{\kappa^2}, u\right) = R^N\left(\frac{k_i}{\rho\kappa}, \rho\kappa L, \frac{t(\rho)}{\rho^2\kappa^2}, u(\rho)\right)$$

Again if we choose ρ such that equation (5) is satisfied we find that R^N is a function of the combination $\rho\kappa L$. Note that using equation (5) we can substitute $t(\rho)$ for $\rho^2\kappa^2$ thus the variable $\rho\kappa L = t(\rho)^{\frac{1}{2}}L$, but as defined $t(\rho) = \xi_L^{-2}$ [7], where ξ_L is the correlation length of the system of size L. The relevant scaling variable is therefore $\frac{L}{\xi_L}$, rather than $\frac{L}{\xi_\infty}$ as it is in the standard formulation. Note that only as $\frac{L}{\xi_L} \to \infty$ will these coincide. We see that equation (5) is equivalent to choosing $\rho\kappa\xi_L = 1$, rather than $\rho\kappa L = 1$ which was the choice of Brézin. Our choice has the advantage of allowing us to probe the situation of divergent correlation length for finite L. The most general statement that we can make about the RG invariant R_N therefore is that

$$R^N = f^N(k_i\xi_L, \frac{L}{\xi_L}, u(\frac{1}{\kappa\xi_L}))$$

This is the general form of the scaling function. Now in the neighbourhood of a critical point, at which $\xi \rightarrow \infty$, $u \rightarrow u^*$, R^N becomes

$$R^N = f^N(k_i\xi_L, \frac{L}{\xi_L})$$

or when the momenta are set to zero, $R^N = f^N(\frac{L}{\xi_L})$. Although there seems to be an apparent dependence on u in the scaling function f explicit $O(\epsilon)$ calculations [6] show that for $L \gg a$, $\xi \gg a$, u is a function of $\frac{L}{\xi_L}$ throughout the crossover regime and so the u dependence is illusory.

That one gets $\frac{L}{\xi_L}$ as the relevant scaling variable can be understood from another point of view in the case of periodic boundary conditions, since the dependence on L enters through the fact that the momenta in the periodic direction now takes discrete values quantized as $k_n = (\frac{2\pi n}{L})$. The dependence of Γ^N on the momenta is of the form $k\xi_L$ which gives $\frac{L}{\xi_L}$ as the expected dependence on L.

Substituting $\rho = \frac{1}{\kappa\xi_L}$ obtained from equation (5) back into equation (4) we obtain the useful expression,

$$\Gamma^N(k_i, L, t, u, \kappa) = (\xi_L)^{N(\frac{d-2}{2})-d} \exp\left(\frac{N}{2}\int_{\frac{1}{\kappa\xi_L}}^{1} \gamma_\phi(x)\frac{dx}{x}\right)\Gamma^N(k_i\xi_L, \frac{L}{\xi_L}, u(\frac{1}{\kappa\xi_L})) \quad (6)$$

This is the general form of the N point function from an RG argument. The novel feature, compared to solving the bulk problem, is that we have found ξ_L to arise rather than ξ_∞.

Let us examine the neighbourhood of a critical point. If, for the problem under discussion there are two potential fixed points, it is of importance to specify which one is in question, this of course depends on the ratio of L to ξ_L. For fixed L there is

only one true fixed point, the reduced one, however, for $\frac{L}{\xi_L} \rightarrow \infty$ with $\xi_L \rightarrow \infty$, the bulk fixed point emerges. For the moment we will not specify which fixed point is in question but treat the neighbourhood of an arbitrary one. In this case γ_ϕ, γ_{ϕ^2} and u approach their fixed point values, which we denote by affixing the superscript *. It is therefore convenient to expand around these values using what are termed "metric factors "to accommodate the fact that one is not exactly at the critical point. These metric factors are slowly varying away from the fixed point, unless one approaches another whereupon they diverge. They express the deviations from exact scaling and play the crucial role of taking us from one fixed point to another.

We express

$$t(\rho) = t\rho^{\gamma^*_{\phi^2}} C_{\phi^2}$$

where

$$C_{\phi^2} = \exp\left(\int_1^\rho (\gamma_{\phi^2}(x) - \gamma^*_{\phi^2})\frac{dx}{x}\right)$$

Similarly

$$Z_\phi(\rho) = Z_\phi \rho^{\gamma^*_\phi} C_\phi$$

where

$$C_\phi = \exp\left(\int_1^\rho (\gamma_\phi(x) - \gamma^*_\phi)\frac{dx}{x}\right)$$

C_{ϕ^2} and C_ϕ are slowly varying metric factors near the fixed point $u = u^*$. Equation (4) therefore becomes

$$\Gamma^N(k_i, L, t, u, \kappa) = (\kappa\rho)^{d-N(\frac{d-2}{2})}\rho^{-\frac{N}{2}\gamma^*_\phi} C_\phi^{-\frac{N}{2}}\Gamma^N(\frac{k_i}{\rho\kappa}, \rho\kappa L, 1, u(\rho), 1) \quad (7)$$

Where ρ is determined by equation (5) and now depends on the one metric factor. We note that when $N = 0$ we are working with the free energy and the metric factor C_ϕ does not enter. There is dependence only on one metric factor, C_{ϕ^2}, in agreement with Privman and Fisher [8]. One subtlety is that in the neighbourhood of the lower dimensional fixed point the dimensions of the fields and the free energy density are different than at the bulk, this implies that $\Gamma^N = (\rho\kappa)^{-\frac{N}{2}(d-d')+(d-d')}\Gamma'^N$ where the prime is used to denote the lower dimensional quantity. With this identification we see that d will correctly become d' the dimension of the reduced system when we consider the reduced fixed point. To obtain the dependence of ρ on t and L we need to examine equation (5) in more detail. In the neighbourhood of the fixed point we find it is of the form.

$$\frac{t}{\kappa^2}\rho^{-2+\gamma^*_{\phi^2}}C_{\phi^2} = 1$$

Now by definition $\gamma^*_{\phi^2} = 2 - \frac{1}{\nu^*}$, where ν^* is the correlation exponent associated with the fixed point under consideration, therefore

$$\rho = \left(\frac{t}{\kappa^2}\right)^{\nu^*} C_2$$

here C_2 is a new metric factor (obtained from $(C_{\phi^2})^{\nu^*}$) containing dependence on L, which caters for the crossover. It is only near a fixed point that we get a scaling variable of the form Lt^{ν^*}, more generally it is $\frac{L}{\xi_L}$. A useful way of parameterizing the crossover is via an effective critical exponent $\nu_{eff} = -\frac{d\ln\xi_L}{d\ln t}$. We can then write

$$\frac{L}{\xi_L} = Le^{\int_\kappa^t \nu_{eff}\frac{dx}{x}}$$

In the limits $\frac{L}{\xi_L} \to \infty$ or $\frac{L}{\xi_L} \to 0$ ν_{eff} is t independent and becomes ν or ν', the bulk or reduced exponent respectively. Substituting back into equation (6), noting $\gamma^*_\phi = \eta^*$, we obtain

$$\Gamma^N(k_i, L, t, u, \kappa) = t^{\nu^*(d-\frac{N}{2}(d-2+\eta^*))}C_\phi^{-\frac{N}{2}}\Gamma^N(k_i t^{-\nu^*}C_2^{-1}, Lt^{\nu^*}C_2, 1, u(t^{\nu^*}C_2), 1) \quad (8)$$

for the N point function in the limit as one of the fixed points is approached, where ν^* and η^* are the associated exponents. It is important to realize that the metric factors can be calculated within the formalism presented.

Sufficently near the bulk fixed point, for the metric factors to be regarded as equal to one, equation (8) can be rewritten as

$$\Gamma^N = \Gamma^N_\infty f(k_i t^{-\nu}, Lt^\nu)$$

where $\Gamma^N_\infty \sim t^{\nu(d-\frac{N}{2}(d-2+\eta))}$, which is the usual form of the scaling relation. If instead we are sufficently near the reduced fixed point we find

$$\Gamma^N = \Gamma'^N_\infty f'(k_i t^{-\nu'}, Lt^{\nu'})$$

where $\Gamma'^N_\infty \sim t^{\nu'(d'-\frac{N}{2}(d'-2+\eta'))}$ is the critical N-point function for the reduced theory, and f' is a finite size scaling function as seen from this perspective.

The main differences between the formulation given here and the standard formulation are: the dependence on $\frac{L}{\xi_L}$ rather than $\frac{L}{\xi_\infty}$, the apparent dependence on u in the crossover region, and the lack of RG invariance of $\frac{\Gamma^N}{\Gamma_\infty}$. The formulations are equivalent when there is no fixed point for finite L. In this case there is only one fixed point for the coupling to be attracted to. We have no proof that the formulations are inequivalent but it seems difficult to believe they are totally equivalent. If the formulations are indeed inequivalent experiment hopefully should be able to determine the correct one.

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