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A String Motivated Approach to the Relativistic Point Particle

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Abstract

Using concepts developed in string theory, Cohen, Moore, Nelson and Polchinski calculated the propagator for a relativistic point particle. Following these authors we extend the technique to include the case of closed world lines. The partition function found corresponds to the Feynman and Schwinger proper time formalisms. We also explicitly verify that the partition function is equivalent to the usual path length action partition function. As an example of a sum over closed world lines, we compute the Euler-Heisenberg effective Lagrangian in a novel way.

Introduction

The quantization of the free relativistic point particle is perhaps the most basic system with constraints studied in physics [1]. In this paper we follow the string motivated techniques developed by Cohen, Moore, Nelson and Polchinski [2] for considering the bosonic point particle. The usual action is proportional to the world line path length and is analogous to the string Nambu-Goto action [3]. Alternatively, a world line metric can be introduced to obtain a more tractable expression in analogy to the Polyakov action [2,3]. These two actions are known to be equivalent at both the classical and quantum level [3,4]. Considering the reparametrization invariance of the Polyakov-like action, as in ref. [2], the partition function can be reduced to a sum over embeddings and a single parameter. This parameter is analogous to the set of modular parameters of a Riemann surface in string theory [5,6]. The dependence of the partition function on this parameter is shown to be different for sums over open and closed particle world lines because of the presence of a diffeomorphism zero mode in the latter case. The parameter in these two cases plays the role of the fictitious "proper time" in the Feynman [7] and Schwinger [8] proper time formalisms. It is explicitly demonstrated as a check on the normalization prescription that this partition function is equivalent to the original path length action partition function. As an example of a process involving closed world lines, we compute the Euler-Heisenberg effective Lagrangian [8] for a boson interacting with a constant external electromagnetic field.

The Relativistic Point Particle

We begin by reviewing the definition of the free relativistic Euclidean point particle Lagrangian (in d dimensions) which is analogous to the Nambu-Goto Lagrangian of string theory [3]. The action S is proportional to the path length (proper time)

$$S[x_\mu] = m\int_0^1 (\dot{x}_\mu^2)^{1/2}\, dt \tag{2.1}$$

where t is a parameterization of the path. In analogy to the Polyakov string we can introduce a metric g(t) along the world line and define an alternative

Polyakov-like action, S_g [3,2]

$$S_g[x_\mu, g] = \frac{1}{2}\int_0^1 \sqrt{g}\,(g^{-1}\dot{x}_\mu^2 + m^2)\,dt \tag{2.2}$$

$$= \frac{1}{2}\int_0^1 (e^{-1}\dot{x}_\mu^2 + m^2 e)\,dt \tag{2.3}$$

where $e = \sqrt{g}$ is the "einbein". It is straightforward to show that

$$S_g[x_\mu, e] \geq S_g[x_\mu, \hat{e}] = S[x_\mu] \tag{2.4}$$

where $\hat{e}$ is the induced einbein

$$\hat{e} = \frac{1}{m}(\dot{x}_\mu^2)^{1/2} \tag{2.5}$$

and hence (2.1) and (2.2) describe the same classical system. Alternatively, solving for the equations of motion one finds the constraint $e = \hat{e}$.

Both S and S_g are reparameterization invariant under $t \longrightarrow s(t)$ with $ds/dt > 0$ where

$$\frac{d}{dt}x_\mu(t) \to \frac{dt}{ds}\frac{dx_\mu}{dt}(t(s))$$

$$e(t) \to \frac{dt}{ds}\, e(t(s)) \tag{2.6}$$

This transformation must however respect the boundary conditions on the world line. Thus for an open path $s(0)=0$, $s(1)=1$ whereas for a closed path $s(t)=s(t+1)$. The parameter $c = \int e(t)dt$ remains invariant and can be used to label diffeomorphically inequivalent metrics. It is analogous to the set of modular parameters of a Riemann surface in the Polyakov string formalism [5,6].

The quantum theories are now defined by the partition functions

$$Z = \int [dx_\mu] \exp(-S) \tag{2.7}$$

$$Z_g = \int [de][dx_\mu] \exp(-S_g) \tag{2.8}$$

We now exploit the reparameterization invariance of S_g to extract a formal diffeomorphic volume factor in (2.8). We change variables from $e(t)$ to $c, f(t)$ where $f(t)$ is the reparameterization which transforms $e(t)$ to c [2]. From (2.6) we find

$$f'(t)\, e(f(t)) = c \tag{2.9}$$

The Jacobian J for this change of variables is most conveniently computed in the target space of einbeins $\{\delta e\}$ [5,2]. We find the equivalent Jacobian for the transformation from δe to $\delta c, \xi$ where $\xi(t) = \delta f(f^{-1}(t))$ is an infinitesimal diffeomorphism vector field i.e. $[d(\delta e)] = J\, d(\delta c)[d\xi]$

We define the normalization for the measure $[d(\delta e)]$ by

$$\int [d(\delta e)] \exp(-\tfrac{1}{2}\|\delta e\|^2) = 1 \tag{2.10}$$

where the invariant norm is

$$\|\delta e\|^2 = \int_0^1 e^{-1}\,\delta e^2\, dt \tag{2.11}$$

From (2.9) we find from ref. [2] that

$$\|\delta e\|^2 = \frac{\delta c^2}{c^2} - \int_0^1 e^3\, \xi\, \Delta \xi\, dt \tag{2.12}$$

where Δ is the Laplacian $\Delta\xi = g^{-1}\frac{d}{dt}(e^{-1}\frac{d}{dt}(e\,\xi))$ The diffeomorphism ξ must obey the boundary conditions $\xi(0) = \xi(1)$ for open paths and $\xi(t)$ periodic for closed paths (the diffeomorphisms of [0,1] and S_1 respectively). For closed paths, ξ = constant corresponding to global rotations introduces a zero mode of Δ. This will imply, as shown below, that different Jacobians occur for closed paths and open paths. The normalization for the ξ integral is

$$\int [d\xi] \exp(-\tfrac{1}{2}\|\xi\|^2) = 1 \tag{2.13}$$

where the invariant norm is

$$\|\xi\|^2 = \int_0^1 e^3\, \xi^2\, dt \tag{2.14}$$

since ξ transforms as a vector.

We now substitute (2.12) into (2.10) and integrate over δc and ξ to obtain J. The δc integral contributes $(2\pi c)^{1/2}$. The integration over ξ depends on the boundary conditions. For open paths we Fourier expand $\xi(t) = \sqrt{2}\sum a_n \sin(n\pi t), n>0$. Then we obtain (using (2.13))

$$\prod_{n>0}\int da_n\left(\frac{c^3}{2\pi}\right)^{1/2}\exp\left(-\frac{1}{2} a_n^2 \cdot \frac{n^2\pi^2}{c^2}\cdot c^3\right)$$

$$= \left[\det_n\left(-\frac{1}{c^2}\frac{d^2}{dt^2}\right)\right]^{-1/2} \sim c^{-1/2} \qquad (2.15)$$

The determinant is easily evaluated by ζ function regularization (see ref. [2]). The reparameterization invariance of (2.12) has been exploited here to choose the gauge $e = c$. The Jacobian J is therefore a constant for open paths.

In the case of closed paths ξ is periodic so we can expand $\xi = \sum b_n \exp(2\pi n i)$ The zero mode b_0 modifies (2.15) so that we find a contribution.

$$\prod_{n}\int db_n\left(\frac{c^3}{2\pi}\right)^{1/2}\exp\left(-\frac{1}{2} b_n^2 \cdot \frac{4n^2\pi^2}{c^2}\cdot c^3\right)$$

$$= \left[\det'_b\left(-\frac{1}{c^2}\frac{d^2}{dt^2}\right)\right]^{-1/2} L\left(\frac{c^3}{2\pi}\right)^{1/2} \sim c^{1/2} \qquad (2.16)$$

where L is a regulator for the b_0 integral. Using ζ function regularization again we find $\det'_b \sim c^2$ since $n<0$ modes also contribute. In this case the Jacobian $J \sim c^{-1}$.

The original partition function (2.8) can now be re-expressed as

$$Z_g = \int_0^\infty dc\, J\, V_D \int[dx_\mu]\exp(-S_g[x_\mu, c]) \qquad (2.17)$$

where $V_D = \int[df]$ is a formal diffeomorphic volume factor which depends on c [9]. An analogous problem arises in the string case where the volume factor depends on the moduli [6]. As for the string case [5,6] we define the physical partition function as

$$Z_{phys} = \int\frac{[de][dx_\mu]}{V_D}\exp(-S_g)$$

$$= \int_0^\infty dc\, J(c)\int[dx_\mu]\exp(-S_g[x_\mu, c]) \qquad (2.18)$$

The appearance of the volume term V_D can be traced to the choice of normalization in (2.10). This point is discussed further below in section 3.

It is satisfying to note that (2.18) now concurs with the Feynman proper time single particle formalism for a bosonic field theory [7] where c is the "proper time" variable. This was illustrated in refs. [2,9] where the correct propagator was calculated. In addition, the Jacobian J introduces the required c dependence for open and closed paths. For closed paths c also plays the role of the proper time in the Schwinger formalism for evaluating determinants [8].

As an example of a sum over closed paths we calculate the effective action for a boson in an external electromagnetic field. The action is modified to include a reparameterization invariant interaction with the external field potential $A_\mu(x_\mu)$ so that

$$Z_A = \int_0^\infty\frac{dc}{c}\int[dx_\mu]\exp\left(-S_g + \int_0^1 \dot{x}_\mu A_\mu dt\right) \qquad (2.19)$$

We can re-express Z_A as

$$Z_A = \int_0^\infty\frac{dc}{c}\, e^{-\frac{m^2}{2}c}\int d^4Y_\mu \langle Y,c|Y,0\rangle \qquad (2.20)$$

where

$$\langle Y,c|Y,0\rangle = \int[dx_\mu]\exp\left(-\int_0^c\left(\frac{1}{2}\dot{x}_\mu^2 + A_\mu\dot{x}_\mu\right)d\tau\right) \qquad (2.21)$$

where $\tau = ct$ is the "proper Euclidean time" and all paths begin and end at Equation (2.21) describes the evolution of a quantum mechanical system with Hamiltonian $H = \frac{1}{2}(P-A)^2$ over a time c. Thus Z_A becomes

$$Z_A = \int_0^\infty \frac{dc}{c}\, e^{-\frac{1}{2}m^2 c}\, \mathrm{Tr}\left(e^{-cH}\right)$$
$$= \mathrm{Tr} \int_0^\infty \frac{dc}{c}\, e^{-\frac{1}{2}((p-A)^2+m^2)c}$$
$$= \mathrm{Tr}\, \log\left((p-A)^2+m^2\right) \tag{2.22}$$

Z_A represents the interaction of a single bosonic particle with an external field and therefore exponentiating we recover the standard result for a bosonic field i.e. $\exp(Z_A) = \det((p-A)^2+m^2)$.

In the case of a constant external field $A_\mu = \frac{1}{2} x_\nu F_{\mu\nu}$, $F_{\mu\nu}$ constant, so that the trace can be explicitly calculated to give the Euler-Heisenberg effective Lagrangian [8]. Alternatively we can compute (2.19) directly since the X integration is Gaussian. This is performed in the appendix.

3. The Quantum Equivalence of S and S_g.

We now demonstrate that the physical partition function of (2.18) is also equivalent to the path length partition function Z. This equivalence will be shown by adopting an explicit Feynman prescription for the einbein path integral. The correctness of the physical partition function prescription of (2.18) will be demonstrated. We begin by stating the result

$$\exp(-S[x_\mu]) = \int [dw]\, \exp(-S_g[x_\mu, e = w^2]) \tag{3.1}$$

where the w measure is normalized to

$$\int [dw]\, \exp\left(-\tfrac{1}{2} m^2 \int w^2 dt\right) = 1 \tag{3.2}$$

corresponding to $x_\mu = 0$ in (3.1). To prove (3.1) we note firstly the useful integral identity

$$\exp(-\sqrt{ab}) = \left(\frac{2a}{\pi}\right)^{1/2} \int_0^\infty dw\, \exp\left(-\tfrac{1}{2}aw^2 - \tfrac{1}{2}bw^{-2}\right) \tag{3.3}$$

where $a, b > 0$. This identity follows from

$$\int_0^\infty dx\, \exp\left(-\tfrac{1}{2}\left(x - \tfrac{\alpha}{x}\right)^2\right) = \sqrt{\tfrac{\pi}{2}}, \quad \alpha \geq 0 \tag{3.4}$$

which can be shown by differentiating with respect to α.

To prove (3.1) consider an arbitrary finite partitioning $t_0, \ldots t_n$ of the interval [0,1] with $t_0=0$, $t_n=1$. We then express the action S as the Riemann sum to find ($\Delta_i t = t_i - t_{i-1}$)

$$\exp(-S) = \lim_{\Delta_i t \to 0} \prod_{i=1}^{n} \exp\left(-m\Delta_i t\left(\left(\frac{\Delta_i x^\mu}{\Delta_i t}\right)^2\right)^{1/2}\right) \tag{3.5}$$

with $\Delta_i x^\mu = x^\mu(t_i) - x^\mu(t_{i-1})$. Now using (3.2) with $a_i = m^2 \Delta_i t$, $b_i = (\Delta_i x^\mu)^2/\Delta_i t$ (3.5) becomes

$$\lim_{\Delta_i t \to 0} \int \prod_{i=1}^{n} dw_i \left(\frac{2m^2\Delta_i t}{\pi}\right)^{1/2} \exp\left(-\frac{\Delta_i t}{2}\left(\left(\frac{\Delta_i x^\mu}{\Delta_i t}\right)^2 w^{-2} + m^2 w_i^2\right)\right),$$

$$= \int [dw]\, \exp\left(-S_g[x_\mu, e = w^2]\right) \tag{3.6}$$

with the normalization of (3.2). The path integral measure [dw] has been explicitly specified by a standard Feynman prescription. Under a reparameterization both S_g and S remain invariant and hence the measure is also invariant. This can be seen directly at the discrete level since a reparameterization corresponds to a repartitioning $\{t_i\} \to \{s_i\}$ of the interval $[0,1]$. The integrand and measure are then clearly invariant under the discrete form of (2.6).

We can now change variables to c and f(t) as before. The Jacobian for the transformation is again computed by working in the tangent space. The invariant norm for δw is

$$\| \delta w^2 \| = \int_0^1 \delta w^2 \, dt = \frac{1}{4} \| \delta e \|^2 \tag{3.7}$$

with normalization

$$\int d[\delta w] \exp\left(-\frac{1}{2} \| \delta w \|^2\right) = \frac{k}{V_0} \tag{3.8}$$

where k is some constant chosen for consistency with the normalization (3.2). Defining the Jacobian J_w by $[d(\delta w)] = J_w \, d(\delta c)[df]$ we find, by an argument similar to that above, that $J_w \sim J(c)/V_0$. Transforming to c and f in (3.1) we obtain

$$Z \propto \int_0^\infty \frac{dc}{c} \frac{J(c)}{V_0} \int [df] \int [dx] \exp(-S_g)$$

$$= Z_{phys} \tag{3.9}$$

Therefore the path length partition function Z is equivalent to the physical Polyakov-like partition function. It is interesting to note that the normalization (3.2) is automatically consistent with the physical definition of the Polyakov partition function of (2.18). This suggests that the natural definition for the normalization of the tangent space measure $[d(\delta e)]$ should be like (3.8).

Finally, we note that since the invariant norms $\| \delta w \|$ and $\| \delta e \|$ are proportional, the Jacobians $J(c)$ and $V_0 J_w$ must also be proportional. Likewise, had we defined Z_g as a sum over all metrics g(t) then, since $\| \delta g \| = 2 \| \delta e \|$, we again find the same Jacobian in transforming to c and f(t).

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Appendix

Euler-Heisenberg Effective Action.

In this appendix we will calculate the effective action from (2.19) in the case where $F_{\mu\nu}$ is constant so that $A_\mu = \frac{1}{2} x_\nu F_{\mu\nu}$. We begin by Fourier expanding the periodic coordinate as $x_\mu = \sum a^n_\mu \exp(2n\pi t i)$. The action of (2.19) becomes (with e=c)

$$S = \frac{1}{2} m^2 c + \sum_n \left(\frac{2n^2\pi^2}{c} a^n_\mu \bar{a}^n_\mu + n\pi i \, \bar{a}^n_\mu a^n_\nu F_{\mu\nu} \right) \tag{A.1}$$

We can now calculate (2.19) by expanding in $F_{\mu\nu}$ using the Feynman rules of Fig. 1.

Expanding (A.1) we find that only even powers of F contribute. In general to $O(F^{2r})$ we find only one connected diagram which contributes

$$\frac{1}{(2r)!} 2^{2r-1} (2r-1)! \; \mathrm{Tr}(F^{2r}) \sum_n \left(\frac{c}{4\pi^2 n^2} \right)^{2r} (n\pi i)^{2r} \tag{A.2}$$

relative to the $O(F^0)$ contribution. The factors are respectively, a symmetrizing factor for 2r identical F sources, a combinatorical factor for th number of ways of connecting 2r vertices to give Fig. 2, a Lorentz index trace and a momentum sum over propagators and vertices. For simplicity we assume a constant electric field only so that $\mathrm{Tr}(F^{2r}) = 2(-1)^r E^{2r}$. Summing over all connected diagrams we find

$$\sum_{r=1}^\infty \frac{1}{r} \left(\frac{Ec}{2\pi} \right)^{2r} \zeta(2r) = \log\left(\frac{Ec/2}{\sin Ec/2} \right) \tag{A.3}$$

where $\zeta(r)$ is the Riemann zeta function. To derive (A.3) we use the relatio

$$\sum_{r=1}^\infty \left(\frac{x}{\pi} \right)^{2r} \zeta(2r) = \frac{1}{2} (1 - x \cot x) \tag{A.4}$$

The contribution from all disconnected diagrams can now be found by exponentiating (A.3). Therefore the partition function (2.19) gives us

$$Z_{EH} = \int_0^\infty \frac{dc}{c}\left(\frac{Ec/2}{\sin(Ec/2)} - 1\right) e^{-m^2 c} \int dY_\mu \langle Y,c|Y,0\rangle_0 \qquad (A.5)$$

where

$$\langle Y,c|Y,0\rangle_0 = \int [dY_r] \exp\left(-\frac{1}{2}\int_0^c \dot{X}_r^2\, d\tau\right) = (2\pi c)^{-d/2} \qquad (A.6)$$

where all paths begin and end at $X_r = Y_r$. Therefore we find that Z_{EH} is (with $c=2s$ and L^d the volume of space)

$$Z_{EH} = \frac{L^d}{(4\pi)^{\frac{d}{2}}} \int_0^\infty \frac{ds}{s^{1+\frac{d}{2}}}\left(\frac{Es}{\sin Es} - 1\right) e^{-sm^2} \qquad (A.7)$$

which is the standard result for a boson [8]. The fermionic result is easily found by also including a spin term which contributes an extra factor to the Lagrangian of $i\, F_{\mu\nu}\sigma_{\mu\nu}$. Tracing over the spin we obtain the fermionic result [8].

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Figure Captions

Fig. 1. Feynman Rules for the propagator and vertex.

Fig. 2. The unique connected Feynman graph in $O(F^{2r})$.

Fig 1

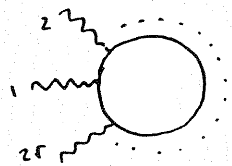

Fig 2