DIAS-STP-80-22

The hard-sphere Boltzmann equation and the Benney moment equations.

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Abstract

The collisionless Boltzmann equation for particles interacting via short-range forces is shown to be equivalent to the Benney equations, which describe long waves in a two-dimensional fluid with a free surface. These systems may also be derived from the nonlinear Schrödinger equation, if multiple scales are introduced and a random phase approximation is employed.

1 Introduction

One of the most peculiar integrable systems of nonlinear evolution equations is that of Benney[1], describing long waves in a perfect fluid with a free surface. Although the system has long been known to possess an infinite set of polynomial conserved densities, there has, as yet, been little progress made towards the solution of its initial value problem.

In the present paper, it will be shown that the Benney equations are equivalent to the Boltzmann equation for a gas of particles interacting via powerful short-range forces (idealised as 'hard spheres'), and also that they may be derived from a crude approximate treatment of slowly modulated periodic solutions of the nonlinear Schrödinger equation.

In section 2 of this paper, the Hamiltonian structure of the collisionless Boltzmann equation is considered. Associated with any finite-dimensional Lie algebra there is a natural Poisson bracket; the Poisson structure considered here is a natural generalisation of this(Kirillov) Poisson bracket, being associated with the single-particle Poisson bracket algebra. The Poisson bracket is also expressed in terms of the moments of the distribution function.

In section 3, the basic results about the Benney equations are reviewed; particular attention is given to the approach of Kupershmidt and Manin[2].

Then, in section 4, it is shown that these equations are equivalent, with some restrictions, to the hard-sphere Boltzmann equation. The methods of Kupershmidt and Manin, described in section 3, are applied to this case, and extended, to determine Riemann invariants for the equations. The method is illustrated by considering the case of a beam of particles.

In section 5, it is shown that multiple scaling of the nonlinear Schrödinger equation, in a random phase approximation, leads to the Benney equations; the results are applied to the case of a slowly modulated monochromatic beam.

Finally, in section 6, various extensions are considered. The relationship between the methods of Kupershmidt and Manin and the scattering problem for the nonlinear Schrödinger equation is described in an appendix.

2 The collisionless Boltzmann equation

The exact N-particle distribution function, F, of a many-body system with Hamiltonian

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2} \sum_{i \neq j} \sum_{j=1}^{N} V_0(x_i - x_j) \tag{2.1}$$

satisfies the equation

$$\frac{\partial F}{\partial t} + p \frac{\partial F}{\partial x} - \frac{\partial V}{\partial x} \frac{\partial F}{\partial p} = 0 \tag{2.2}$$

where

$$V(x) = \int_{-\infty}^{\infty} \int_{0<|x'-x|<\infty} V_0(x-x')\, F(x', p', t)\, dx'\, dp' \tag{2.3}$$

and F is given by

$$F(x,p,t) = \sum_{i=1}^{N} \delta(x - x_i(t))\, \delta(p - p_i(t)) \tag{2.4}$$

If we average over an ensemble of exact distributions, then (2.2) becomes

$$\frac{\partial \bar{F}}{\partial t} + p \frac{\partial \bar{F}}{\partial x} = \frac{\partial}{\partial p} \iint \frac{\partial}{\partial x}(V_0(x-x'))\, \left(\overline{F(x,p)\, F(x',p')}\right) dx'\, dp' \tag{2.5}$$

If we then ignore the correlations between the particles, which is permissible as N becomes large, we find that the averaged distribution function, denoted subsequently by f(x,p,t), satisfies the system of equations:

$$\frac{\partial f}{\partial t} + p\frac{\partial f}{\partial x} - \frac{\partial V}{\partial x}\frac{\partial f}{\partial p} = 0$$

$$V = \int_{-\infty}^{\infty}\int_{0<|x'-x|<\infty} V_0(x-x')\, f(x',p')\, dx'dp' \tag{2.6}$$

This system has a first integral, which is the average of (1):

$$\mathcal{H} = \iint \tfrac{1}{2} p^2 f(x,p)\, dx dp + \tfrac{1}{2}\iiiint_{|x-x'|>0} f(x,p)\, V_0(x-x')\, f(x',p')\, dx dp\, dx' dp' \tag{2.7}$$

This may be interpreted as the Hamiltonian of (6), provided we can find a suitable Poisson bracket between functionals of f. An appropriate bracket is given by

$$\{\mathcal{J}, \mathcal{K}\} = \iint f\left[\frac{\delta \mathcal{J}}{\delta f}, \frac{\delta \mathcal{K}}{\delta f}\right] dx dp \tag{2.8}$$

where $[\ ,\ ]$ is the usual single-particle Poisson bracket:

$$[J, K] = \frac{\partial J}{\partial p}\frac{\partial K}{\partial x} - \frac{\partial J}{\partial x}\frac{\partial K}{\partial p} \tag{2.9}$$

If f is the exact distribution function, then (2.8) is the natural N-particle Poisson bracket.

Mathematically, (2.8) may be considered as an analogue of the Kirillov Poisson bracket, which is defined for finite-dimensional Lie algebras [3]. Here, however, the relevant Lie algebra is the space of C^{∞} functions on phase

space together with the single-particle Poisson bracket (2.9). Since this algebra is infinite-dimensional, the construction is less natural than that of Kirillov; however (2.8) makes sense for reasonable physical cases.

The moments of f , defined by

$$A^{(n)}(x) = \int p^n f(x,p)\, dp \tag{2.10}$$

are of some interest; the Hamiltonian (2.7), for example, may be written

$$\mathcal{H} = \int \tfrac{1}{2} A^{(2)} dx + \tfrac{1}{2}\iint_{|x-x'|>0} A^{(0)}(x)\, V(x-x')\, A^{(0)}(x')\, dx dx' \tag{2.11}$$

The Poisson bracket (2.8) of two functionals of the moments alone is given by

$$\{\mathcal{J}, \mathcal{K}\} = \int \sum_{m,n=0}^{\infty}\sum^{\infty} \frac{\delta \mathcal{J}}{\delta A^{(m)}}\left(m A^{(n+m-1)}\frac{\partial}{\partial x} + \frac{\partial}{\partial x} A^{(n+m-1)} n\right)\frac{\delta \mathcal{K}}{\delta A^{(n)}}\, dx \tag{2.12}$$

This form of Poisson bracket also arises in the study of the Benney equations, which will be discussed in the next section.

A particular case of (2.6), with which we will be concerned below, is the 'hard-sphere' case. Here the potential $V_0(x - x')$ tends to a δ -function; if we were concerned with exact distributions, the resulting equation would indeed describe perfectly elastic particles. However, for the neglect of correlations in passing from (2.4) to (2.6) to be valid, the range of the force must decrease as the number of particles increases, in order that each particle always interacts with many others, and only then may inter-particle correlations be neglected. Only in this sense is it reasonable to talk about a 'collisionless' hard-sphere gas. The Boltzmann equation for this system is

$$\frac{\partial f}{\partial t} + p\frac{\partial f}{\partial x} - \int_{-\infty}^{\infty} \frac{\partial f}{\partial x}\, dp\, \frac{\partial f}{\partial p} = 0 \tag{2.13}$$

and its Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\int A^{(2)} + \left(A^{(0)}\right)^2 dx \tag{2.14}$$

One very reasonable restriction of any Boltzmann equation is to consider a beam of particles:

$$f(x,p) = h(x)\,\delta(p - u(x)) \tag{2.15}$$

The Poisson bracket (2.12) reduces in this case to:

$$\{\mathcal{G}, \mathcal{K}\} = \int \frac{\delta \mathcal{G}}{\delta h}\frac{\partial}{\partial x}\frac{\delta \mathcal{K}}{\delta u} + \frac{\delta \mathcal{G}}{\delta u}\frac{\partial}{\partial x}\frac{\delta \mathcal{K}}{\delta h}\, dx \tag{2.16}$$

which also arises in the study of the Benney equations.

3 The Benney equations

Benney, by expanding the Euler equations for an incompressible fluid in a small parameter, the ratio of the depth of the water to the length of the waves, obtained the following system of equations describing long waves beneath a free surface:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \int_0^y \frac{\partial u}{\partial x}\, dy' \frac{\partial u}{\partial y} + \frac{\partial h}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^h u\, dy = 0 \tag{3.1}$$

He also introduced the moments

$$A^{(n)}(x) = \int_0^h u^n(x,y)\, dy \tag{3.2}$$

and showed that

$$A^{(n)}_t + A^{(n+1)}_x + n\, A^{(0)}_x\, A^{(n-1)} = 0 \tag{3.3}$$

He then showed that the equations (3.3) possess an infinite set of conserved densities; Miura[4] extended his argument and established that these conservation laws could be written in local form. Kupershmidt and Manin discuss the Hamiltonian structure of (3.3); the Poisson bracket is the same as (2.12).

In this formalism, the Hamiltonian of (3.3) is the same as (2.14); indeed the moments of the hard-sphere Boltzmann equation do satisfy the system (3.3).

The conservation laws are most easily derived by constructing a generating function for the moments;

$$\Phi(\lambda) = \int_0^h \frac{dy}{u+\lambda} \tag{3.4}$$

and then solving the following equation for μ ;

$$\mu = \lambda - \Phi(\mu)$$

$$\mu = \lambda + O(\lambda^{-1}) \tag{3.5}$$

Here μ is uniquely defined as a formal series for large λ ; if u is bounded, this will converge to an analytic function for $|\lambda|$ large enough. Then μ is shown to satisfy a conservation equation, and the coefficients in the series expansion of μ are conserved densities. On imposing the restriction on (3.1) that the flow should be irrotational, that is, that u should be independent of y , the following 'reduced' system is obtained

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial h}{\partial x} = 0$$
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0 \tag{3.6}$$

This is derived from the Hamiltonian density

$$\mathcal{H} = \tfrac{1}{2} h u^2 + \tfrac{1}{2} h^2 \tag{3.7}$$

together with the Poisson bracket (2.16). The system (3.6) is also obtainable from the hard-sphere Boltzmann equation (2.13) if a beam of particles is considered.

4 The hard-sphere gas

Since the apparently distinct systems (2.13) and (3.1) generate the same moment equations, it is worth checking to see whether they can be transformed into one another directly. Comparison of the two definitions of $A^{(n)}$ suggests the introduction of the new variable

$$y = \int_{-\infty}^{u} f(x,p,t)\, dp \tag{4.1}$$

If we now regard u as a function of y , rather than vice versa, we indeed obtain the system (3.1) from the equation (2.13), by means of a (reverse) hodograph transformation[5]. However, there are several differences; in the fluid system it would be quite reasonable for $\frac{\partial u}{\partial y}$ to change sign at some point in the fluid; this would,however correspond to a singularity of the distribution f . On the other hand, it is perfectly reasonable to expect f , considered as a function of p , not to have compact support, but merely to decay rapidly as $|p| \to \infty$. In the fluid model this would correspond to infinite vorticity at the boundaries $y \to 0$ and $y \to h$. With these reservations, the systems may be regarded as equivalent. The methods of Kupershmidt and Manin may be applied, with few modifications, to the Boltzmann equation. The generating function for the moments becomes

$$\Phi(\lambda) = P\int_{-\infty}^{\infty} \frac{f(x,p)}{\lambda + p}\, dp \tag{4.2}$$

which is essentially the Hilbert transform of f ; since f is rapidly decreasing, and it is predominantly the asymptotic behaviour of Φ for large λ which is of interest, the manner in which we treat the pole of the integrand is irrelevant, though the principal part seems the most reasonable choice. The variable μ is defined in the same way as before:

$$\mu = \lambda - \Phi(\mu) \tag{4.3}$$

Clearly

$$\frac{\partial \lambda}{\partial \mu} = 1 + \frac{\partial}{\partial \mu} P\int_{-\infty}^{\infty} \frac{f(x,p)}{\mu + p}\,dp = 1 - P\int_{-\infty}^{\infty} \frac{\frac{\partial f}{\partial p}(x,p)\,dp}{\mu + p} \tag{4.4}$$

This derivative is positive for large real μ, and thus μ is a well defined implicit function for large real λ. In fact the whole of the curve defined by (4.3) is of importance, although it is the asymptotic behaviour of μ for large λ which gives the conservation laws.

Now, differentiating with respect to time in (4.3),

$$\frac{\partial \mu}{\partial t}\left(1 - P\int_{-\infty}^{\infty} \frac{\frac{\partial f}{\partial p}}{\mu + p}\,dp\right) = -P\int_{-\infty}^{\infty} \frac{\frac{\partial f}{\partial t}\,dp}{\mu + p} + \frac{\partial \lambda}{\partial t}$$

$$= P\int_{-\infty}^{\infty} \frac{p\frac{\partial f}{\partial x}}{\mu + p}\,dp - \int_{-\infty}^{\infty} \frac{\partial f}{\partial x}\,dp\; P\int_{-\infty}^{\infty} \frac{\frac{\partial f}{\partial p}}{\mu + p}\,dp + \frac{\partial \lambda}{\partial t}$$

$$= \int_{-\infty}^{\infty} \frac{\partial f}{\partial x}\,dp\left(1 - P\int_{-\infty}^{\infty} \frac{\partial f/\partial p}{\mu + p}\,dp\right)$$
$$- \mu P\int_{-\infty}^{\infty} \frac{\partial f/\partial x}{\mu + p}\,dp + \frac{\partial \lambda}{\partial t} \tag{4.5}$$

Similarly, we get

$$\frac{\partial \mu}{\partial x}\left(1 - P\int_{-\infty}^{\infty} \frac{\frac{\partial f}{\partial p}}{\mu + p}\,dp\right) = \frac{\partial \lambda}{\partial x} - P\int_{-\infty}^{\infty} \frac{\frac{\partial f}{\partial x}}{\mu + p}\,dp \tag{4.6}$$

Thus

$$\left(\frac{\partial \mu}{\partial t} - \frac{\partial}{\partial x}\left(\frac{\mu^2}{2} + A^{(0)}\right)\right)\left(1 - P\int_{-\infty}^{\infty} \frac{\frac{\partial f}{\partial p}}{\mu + p}\,dp\right) = \frac{\partial \lambda}{\partial t} - \mu\frac{\partial \lambda}{\partial x} \tag{4.7}$$

There are two possibilities now; we can either work with a fixed value of λ, giving a conservation equation for μ, or, alternatively, we may consider the points (μ_i, λ_i) such that the derivative (4.4) vanishes, and then the $\{\lambda_i\}$ are the Riemann invariants[6] of the system, while their associated $\{(-\mu_i)\}$ are the group velocities with which they propagate.

By way of illustration, let us investigate the reduced system, considered, in the Boltzmann picture, at the end of section 2, and in the fluid case, at the end of section 3:

$$f = h(x,t)\,\delta(p - u(x,t))$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial h}{\partial x} = 0 \tag{4.8}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$$

Clearly

$$\Phi(\lambda) = \frac{h}{u + \lambda} \tag{4.9}$$

Hence the curve (4.3) is defined by:

$$\mu = \lambda - h/_{u+\mu} \tag{4.10}$$

This gives us that the Riemann invariants and group velocities are:

$$\lambda_\pm = -u \pm 2\sqrt{h}$$
$$\mu_\pm = -u \pm \sqrt{h} \tag{4.11}$$

Hence

$$\frac{\partial \lambda_\pm}{\partial t} = \frac{1}{4}\left(3\lambda_\pm + \lambda_\mp\right)\frac{\partial \lambda_\pm}{\partial x} \tag{4.12}$$

If one of the invariants is constant, then only the constant solution will remain single-valued for all time.

This approach bears some similarities to the results of Flaschka, Forest, and McLaughlin[7], who consider slowly modulated multiply periodic solutions of the KdV equation. Such multiply periodic solutions are related to hyperelliptic Riemann surfaces; the modulation equations may be expressed invariantly by requiring a differential form on the surface to vanish. By expanding this form at the point at infinity, one obtains the conservation laws; by expanding near the branch points, one finds that the branch points are themselves the Riemann invariants.

Just as the study of the deformation of a complex curve yields different descriptions of the modulation equations for the n-gap KdV solutions, so are the Benney equations described by studying the deformation of the curve defined by (4.3). This similarity is not fortuitous; the Benney equations may be derived by considering the deformations on long space and time scales of multiply periodic solutions of another integrable system, the nonlinear Schrödinger equation.

5 The nonlinear Schrödinger equation

The equation

$$i\frac{\partial\psi}{\partial t} + \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} = |\psi^2|\psi \tag{5.1}$$

has an infinite set of conserved densities, polynomial in ψ, ψ^* and their derivatives. The first few are:

$$\begin{aligned} H^{(0)} &= |\psi^2| \\ H^{(1)} &= -\frac{i}{2}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right) \\ H^{(2)} &= \left|\frac{\partial\psi}{\partial x}\right|^2 + |\psi|^4 \\ H^{(3)} &= \frac{i}{2}\left(\psi^*\frac{\partial^3\psi}{\partial x^3} - \psi\frac{\partial^3\psi^*}{\partial x^3}\right) - \frac{3i}{2}|\psi^2|\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right) \end{aligned} \tag{5.2}$$

These may be compared with the first few conserved densities of the Benney moment equations:

$$\begin{aligned} H^{(0)} &= A^{(0)} \\ H^{(1)} &= A^{(1)} \\ H^{(2)} &= A^{(2)} + \left(A^{(0)}\right)^2 \\ H^{(3)} &= A^{(3)} + 3A^{(0)}A^{(1)} \end{aligned} \tag{5.3}$$

Thus one is tempted to make the identification

$$A^{(n)} \sim \psi^*(-i\partial)^n\psi \tag{5.4}$$

However, it is fairly obvious that this is not very useful; if, though, we consider slowly varying multiply periodic solutions, we may average over many periods, and then the $A^{(n)}$ will be moments of the (suitably normalised) power spectrum of ψ. Thus we take, for some appropriate L,

$$A^{(n)}(x) = \frac{1}{2L}\int_{x-L}^{x+L} \psi^* \left(-i\frac{\partial}{\partial x}\right)^n \psi \, dx'$$

$$= \int_{-\infty}^{\infty} k^n f(k)\, dk \tag{5.5}$$

In section 2, to derive the Boltzmann equation, it was necessary to ignore correlations between particles; analogously, we here replace $|\psi^2|$ on the right of (5.1) by its average, giving:

$$i\frac{\partial \psi}{\partial t} + \frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} = A^{(0)}(x)\,\psi \tag{5.6}$$

Now $A^{(0)}$ is slowly varying, and it is thus appropriate to introduce multiple length and time scales:

$$x_0 = x \qquad x_1 = \epsilon x \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial x_0} + \epsilon\frac{\partial}{\partial x_1}$$

$$t_0 = t \qquad t_1 = \epsilon t \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon\frac{\partial}{\partial t_1} \tag{5.7}$$

where ϵ is the ratio of a typical wavelength to the length scale on which the $A^{(n)}$ vary. L, introduced in (5.5), should be intermediate between these two scales. Now (5.6) becomes, in zero order,

$$i\frac{\partial \psi}{\partial t_0} + \frac{1}{2}\frac{\partial^2 \psi}{\partial x_0^2} = A^{(0)}(x_1, t_1)\,\psi \tag{5.8}$$

which is linear in ψ, with dispersion relation

$$\omega = \frac{1}{2}k^2 + A^{(0)}(x_1, t_1) \tag{5.9}$$

It is well known that a normal mode of (5.8) will propagate on the (x_1, t_1) scale like a particle with Hamiltonian ω, depending on x_1 and its conjugate momentum, k. Hence the power spectrum, $f(x_1, k, t_1)$, must satisfy

$$\frac{\partial f}{\partial t_1} + k\frac{\partial f}{\partial x_1} - \frac{\partial A^{(0)}}{\partial x_1}\frac{\partial f}{\partial k} = 0 \tag{5.10}$$

which is just (2.13).

Although the approximation which gives (5.6) is generally very crude, there is one case where it is quite accurate; this is when we consider solutions of the form

$$\psi = \sqrt{h(x_1, t_1)}\; e^{i\left(u(x_1,t_1)x_0 - \frac{1}{2}\left(u^2(x_1,t_1) + h(x_1,t_1)\right)t_0\right)} \tag{5.11}$$

The modulation of such solutions is adequately described, to first order, by the reduced Benney equations (3.6). Since the Benney system has solutions which become multiply valued in finite time, one may conclude that the ansatz (5.11) breaks down in general, and that slowly-modulated single-gap solutions will be connected, via 'shock' regions, to 3-gap solutions. The final description of such a solution should be quite complicated.

6 Conclusions

This paper has had several main objects. The first was to describe the Hamiltonian structure of the Boltzmann equation, and to set up a framework for discussing more general integrable many-body problems. The form of the Poisson bracket (2.12) suggested the relationship between the hard-sphere gas and the Benney equations, and the second object of this paper was to exploit this connection to study the Benney equations further. Finally, the relationship between the nonlinear Schrödinger and Benney systems tends to suggest why the latter equations should be integrable; how they might be integrated, of course, is a much more difficult problem. It is perhaps significant that the two-dimensional Benney system arises from multiple scaling of the one-dimensional NLS.

An obvious extension of the present work is to consider other integrable systems; the Calogero-Moser gas[8], whose particles interact via an inverse-square potential, can also be studied by the methods described here[9]. A related approach is to consider some integrable differential equation, such as the derivative NLS

$$i\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} = 2i|\psi^2|\frac{\partial\psi}{\partial x} \qquad (6.1)$$

and, by applying the methods of section 5, generate an integrable mechanical system and the associated moment equations[10].

Acknowledgements

I would like to thank Allan Fordy for many useful discussions, and for his helpful comments on an earlier draft of this paper.

Appendix

The nonlinear Schrödinger equation (5.1) is an isospectral flow of the self-adjoint operator

$$L = \begin{pmatrix} i\frac{\partial}{\partial x} & \psi \\ \psi^* & -i\frac{\partial}{\partial x} \end{pmatrix} \qquad (A.1)$$

We consider the eigenvalue problem (where $\underset{\sim}{\phi} = (\phi_1, \phi_2)^T$),

$$L\underset{\sim}{\phi} = \zeta\underset{\sim}{\phi} \qquad (A.2)$$

Taking the Fourier transform, we get

$$\begin{aligned} (\zeta - k)\hat{\phi}_1(k) &= \int \hat{\psi}(k')\hat{\phi}_2(k-k')\,dk' \\ (\zeta + k)\hat{\phi}_2(k) &= \int \hat{\psi}^*(k')\hat{\phi}_1(k-k')\,dk' \end{aligned} \qquad (A.3)$$

Eliminating $\hat{\phi}_1$ in the second of (A.3) gives

$$(\zeta + k)\hat{\phi}_2(k) = \iint \frac{\hat{\psi}^*(k')\hat{\psi}(k'')}{\zeta - k + k'}\phi_2(k-k'-k'')\,dk'dk'' \qquad (A.4)$$

In the random phase approximation, we replace $\hat{\psi}^*(k')\hat{\psi}(k'')$ by the expression $f(k')\delta(k'+k'')$, where $f(k')$ is the power spectrum, normalised as in section 5. Then

$$(\zeta + k)\hat{\phi}_2(k) = \int \frac{f(k')}{\zeta - k + k'}\,dk'\,\hat{\phi}_2(k) \qquad (A.5)$$

Hence

$$\zeta - k = 2\zeta - \int \frac{f(k')}{(\zeta - k) + k'}\, dk' \tag{A.6}$$

Replacing $(\zeta - k)$ by μ, and 2ζ by λ, we get

$$\mu = \lambda - \int \frac{f(k')}{\mu + k'}\, dk' \tag{A.7}$$

which is the same as equation (4.3).

The relationship between scattering transforms (nonlinear Fourier transforms), and the method of Kupershmidt and Manin, which may be regarded as a nonlinear Hilbert transform, is certainly worth investigating further.

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