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Conformal Properties of the BPST

Instantons of the generalised Yang-Mills System

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ABSTRACT: A manifestly $O(4p + 1)$ invariant formulation of generalised Yang-Mills (GYM) theory on $S^{4p}\{r|r.r = 1\} \subset R^{4p+1}$ is given, and the corresponding BPST instantons and anti-instantons are shown to be solutions of the equations of motion.

Our main purpose in this note is to complement a paper by Grossman, Kephart and Stasheff[1]. These authors[1] have found a spherically symmetric solution to the following duality equations in 8-dimensions

$$F \wedge F = {}^*(F \wedge F) \qquad (1)$$

This solution is the analogue of the BPST instanton[2] in 4-dimensions, and (1) is the $p = 2$ member of the hierarchy of duality relations in $4p$-dimensions

$$F(2p) = \pm {}^*F(2p) \qquad (2a,b)$$

where

$$F(2p) = F \wedge \cdots \wedge F \quad (2p \text{ times})$$

These are the duality relations pertaining to the generalised Yang-Mills (GYM) system[3)4] with action

$$\int \mathrm{tr}\, F(2p)^2 \, d^{4p} x \ . \qquad (3)$$

The solutions of (2) yield a hierarchy[3] of BPST instantons.

Now an important property of the BPST instanton field configuration is its behaviour under conformal transformations[5], as was stressed and exploited by Jackiw and Rebbi[6]. These authors noted that the BPST instanton was actually invariant under the action of an $O(5)$ subgroup of the conformal group. Then they gave a manifestly $O(5)$-invariant formulation for the Yang-Mills (YM) system, by incorporating the SU(2) instanton and anti-instanton fields in an $O(4)$-algebra valued field on $S^4\{r|r.r = 1\} \subset R^4$. This formulation does generalise to dimension $4p$, which we will give below.

In dimension-8 Grossman et al [1] attempted to derive the solutions of (1) from the Euler-Lagrange equations of the SO(8) YM system on $S^8\{r|r.r = 1\} \subset R^9$ and $\underline{\text{not}}$ from the $p = 2$ GYM system, whose instantons we

know[3] are the solutions to (1). Later in an Erratum[7], they pointed out that the GYM action (3) (with p = 2) should replace the YM action in their work, but did not give the details of the formulation in this case, that Jackiw and Rebbi[6] have given in dimension 4. This is precisely what we present below.

For arbitrary p, the BPST instanton and anti-instanton connections[3] can be expressed, respectively, in the following forms

$$A^{\mu}(x) = \frac{x^2}{1+x^2} \; g^{-1}\partial_{\mu} g \tag{4a}$$

$$\tilde{A}^{\mu}(x) = \frac{x^2}{1+x^2} \; \tilde{g}^{-1}\partial^{\mu}\tilde{g} \tag{4b}$$

$$g = (x\cdot x)^{-\frac{1}{2}} \; x\cdot\Sigma \quad , \quad \Sigma^{\mu} = \frac{1}{2}(1+\Gamma_{4p+1})\,\Gamma^{\mu} \tag{5a}$$

$$\tilde{g} = (x\cdot x)^{-\frac{1}{2}} \; x\cdot\tilde{\Sigma} \quad , \quad \tilde{\Sigma}^{\mu} = \frac{1}{2}(1-\Gamma_{4p+1})\,\Gamma^{\mu} \tag{5b}$$

where Γ^{μ} are the $2^{2p} \times 2^{2p}$ Γ-matrices in 4p-dimensions, and Γ_{4p+1} is the corresponding chirality matrix.

The corresponding field strengths are then given by

$$F^{\mu\nu}(x) = \frac{4}{(1+x^2)^2} \; \Sigma^{\mu\nu} \tag{6a}$$

$$\tilde{F}^{\mu\nu}(x) = \frac{4}{(1+x^2)^2} \; \tilde{\Sigma}^{\mu\nu} \tag{6b}$$

where

$$\Sigma^{\mu\nu}(x) = -\frac{1}{4}\left(\frac{1+\Gamma_{4p+1}}{2}\right)\,[\Gamma^{\mu},\Gamma^{\nu}] \tag{7a}$$

$$\tilde{\Sigma}^{\mu\nu}(x) = -\frac{1}{4}\left(\frac{1-\Gamma_{4p+1}}{2}\right)\,[\Gamma^{\mu},\Gamma^{\nu}] \tag{7b}$$

are spinor-representations of SO(4p), and for p = 1 they are self and antiself-dual, respectively.

The 2p-forms used in (2a,b) then take the following forms

$$F_{\mu_1\cdots\mu_{2p}}(x) = \frac{4^p}{(1+x^2)^{2p}} \; \Sigma_{\mu_1\cdots\mu_{2p}} \tag{8a}$$

$$\tilde{F}_{\mu_1\cdots\mu_{2p}}(x) = \frac{4^p}{(1+x^2)^{2p}} \; \tilde{\Sigma}_{\mu_1\cdots\mu_{2p}} \tag{8b}$$

where $\Sigma_{(p)}$ and $\tilde{\Sigma}_{(p)}$ are totally antisymmetrised products of $\Sigma_{\mu}(\tilde{\Sigma}_{\mu})$ and $\tilde{\Sigma}_{\mu}(\Sigma_{\mu})$ respectively, satisfying the following duality relations

$$\Sigma_{\mu_1\cdots\mu_{2p}} = {}^*\Sigma_{\mu_1\cdots\mu_{2p}} \tag{9a}$$

$$\tilde{\Sigma}_{\mu_1\cdots\mu_{2p}} = -{}^*\tilde{\Sigma}_{\mu_1\cdots\mu_{2p}} \tag{9b}$$

Then (8a,b) manifestly satisfy (2a,b).

To incorporate the instanton and anti-instanton field configurations (8a,b), we consider the co-ordinate inversion transformations

$$A^{\mu}(x) \longrightarrow \bar{A}^{\mu}(x) = \frac{1}{x^2} \; I^{\mu\nu}(x)\,A_{\nu}(\mathit{1/x}) \;, \tag{10}$$

$$I^{\mu\nu}(x) = \delta^{\mu\nu} - 2\,x^{\mu}x^{\nu}/x^2 \;.$$

Under (10), the 2p-forms (8a,b) transform as

$$F_{\mu_1\cdots\mu_{2p}}(x) \to \bar{F}_{\mu_1\cdots\mu_{2p}}(x) = \frac{1}{x^{4p}} \; I^{\mu_1\nu_1}(x)\cdots I^{\mu_{2p}\nu_{2p}}(x)\,F_{\nu_1\cdots\nu_{2p}}(\mathit{1/x}) \;, \tag{11}$$

whence it can be shown easily that (2a,b) result in the duality relations

$$\bar{F}(2p) = \mp {}^*\bar{F}(2p) \;. \tag{12a,b}$$

From (2a,b) and (12a,b) one concludes that under inversion, which belongs to the conformal group, instantons and anti-instantons are interchanged. Also, since the topological charge of these instantons[3] are given by the 2p-th Chern-Pontryagin integrals, and the latter are integrals over the densities

$$\text{tr}\;\; F(2p) \wedge F(2p) \;,$$

it is clear that a q = 1 BPST instanton will simply go over to a q = 1 BPST anti-instanton, under the inversion (10).

This leads us to consider the composite field strength $F^{\mu\nu} + \tilde{F}^{\mu\nu}$ in (6a,b). We denote this field simply as

$$F^{\mu\nu}(x) = \frac{4}{(1+x^2)^2} \; \Gamma^{\mu\nu} \tag{13}$$

$$\Gamma^{\mu\nu} = -\tfrac{1}{4} \left[\Gamma^{\mu}, \Gamma^{\nu}\right] . \tag{14}$$

$\Gamma^{\mu\nu}$ in (14) is simply the representation $(\bar{\Sigma}^{\mu\nu} \mathbf{r} \check{\Sigma}^{\mu\nu})$ of SO(4p), and in (13) incorporates the $q = +1$ and $q = 1$ fields in (6a,b). Under an inversion transformation, these $q = \pm 1$ fields are interchanged. Under a full conformal transformation (inversion-translation-inversion) $F^{\mu\nu}$ is not invariant.

It can however be shown simply by adapting the demonstration given in ref.[6] (for $p = 1$), that for any p the field (13) is invariant under an O(4p + 1) subgroup of the conformal group O(4p + 1,1). The O(4p + 1) group in question is generated by $(M^{\mu\nu}, R^{\mu})$, where $M^{\mu\nu}$ are the generators of 4p-dimensional rotations and

$$R^{\mu} = \tfrac{1}{2}\left(P^{\mu} + K^{\mu}\right) \tag{15}$$

is a combination of the translation and (special) conformal transformation generators P^{μ} and K^{μ}, respectively, in 4p-dimensions. The group generated by (15), for the $p = 1$ case, has been previously considered by Adler[8] and Fubini[9].

Following Jackiw and Rebbi[6], and in turn Adler[8] who first formulated (euclidean) electrodynamics on S^4, we proceed to generalise the formulation[6] of the $p = 1$ GYM (i.e. YM) system on S^4, to the case of general p on S^{4p} $\{r|r.r = 1\} \subset R^{4p+1}$.

We first summarise the necessary definitions and formulas[6][8] appropriate to any p. The R^{4p+1} coordinates $r_a = r_\mu, r_{4p+1}$ are defined by

$$r_\mu = \frac{2x_\mu}{1+x^2} \quad , \quad r_{4p+1} = \frac{1-x^2}{1+x^2} , \tag{16}$$

and the O(4p + 1)-algebra valued gauge connections $\hat{A}_a$ by

$$\frac{1+x^2}{2} A_\mu = \hat{A}_\mu - x_\mu \hat{A}_{4p+1}$$

$$\hat{A}_\mu = \frac{1+x^2}{2} A_\mu - x_\mu x_\nu A_\nu \tag{17}$$

$$\hat{A}_{4p+1} = -x_\mu A_\mu .$$

The (antihermitian) representations of the O(4p + 1) that we use are

$$\Gamma^{ab} = \left(\Gamma^{\mu\nu}, \tfrac{1}{2}\Gamma^{\mu}\right) \tag{18}$$

with $\Gamma^{\mu\nu}$ given by (14). The gauge transformation formulas[6][7] for the connections $\hat{A}_a$ are

$$\hat{A}_a \xrightarrow{u} \; \mathcal{U}^{-1}\hat{A}_a \mathcal{U} + \mathcal{U}^{-1} r_b L_{ba} \mathcal{U} , \tag{18a}$$

where L_{ab} are the following "angular momentum" operators

$$L_{ab} = r_a \, \partial/\partial r_b - r_b \, \partial/\partial r_a$$

$$L_{\mu\nu} = x_\mu \, \partial/\partial x_\nu - x_\nu \, \partial/\partial x_\mu \tag{19}$$

$$L_{4p+1,\mu} = x_\mu x^\nu \, \partial/\partial x^\nu - (1-x^2) \, \partial/\partial x^\mu .$$

In addition, these connections obey the constraint[6][7]

$$r_a \hat{A}_a = 0 . \tag{20}$$

The curvature, or gauge covariant field strength, corresponding to $\hat{A}_a$ is

$$\hat{F}_{abc} = L_{ab}\hat{A}_c + r_a \left[\hat{A}_b, \hat{A}_c\right] + \text{cyclic permutations in } a,b,c. \tag{21}$$

For $p = 1$, the invariant action is[6]

$$\int d\Omega \; \mathrm{tr} \; \hat{F}_{abc}^2 , \tag{22}$$

which can be shown, after converting the volume element $d\Omega$ on S^4 to d^4x on R^4, to be proportional to the $p = 1$ GYM (i.e. YM) action. Then (22) was subjected[6] to the variational principle, whereby the $p = 1$ BPST solutions (instanton and anti-instanton) (13) were found without reference to the self-duality relations (2a,b). This was precisely what Grossman et al[1][7] set out to do for $p = 2$, and which we present below.

To construct the action (22) for $p = 1$, all one needs is the definition (21) for the 3rd rank totally antisymmetric field strength $\hat{F}(3) = \hat{F}_{abc}$. For arbitrary p, we find that the generalisation of (22), leading to the corresponding GYM action (3) on R^{4p}, is

$$\int d\Omega \; \text{tr} \; \hat{F}(2p+1)^2 , \tag{23}$$

where $d\Omega$ is again the volume element of the S^{4p+1} coordinates, and $\hat{F}(2p+1) = \hat{F}_{a_1 \cdots a_{2p+1}}$ is a $(2p+1)$-th rank totally antisymmetric gauge covariant field strength on $R^{4p+1} \supset S^{4p}\{r|r.r = 1\}$.

The definition of $\hat{F}(2p+1)$ can be made inductively from $p = 2$ onwards. Therefore, in addition to $\hat{F}(3)$ given by (21), the only other gauge covariant field strength we need is a 2nd rank antisymmetric tensor field $\hat{F}_{ab} = \hat{F}(2)$, on R^{4p+1}, which should not be confused with the curvature field $F_{\mu\nu} = F(2)$ on R^{4p}.

This new field strength $\hat{F}(2)$ is

$$\hat{F}_{ab} = r_c L_{c[a} \hat{A}_{b]} + [\hat{A}_a, \hat{A}_b] - r_{[a} \hat{A}_{b]} . \tag{24}$$

Once we construct $\hat{F}(5)$ for $p = 2$, that procedure then can be systematically extended to any p. Here we will present the $p = 2$ case in detail.

For $p = 2$, from (21) and (24), we define $\hat{F}(5)$ as

$$\hat{F}_{abcde} = \{ \hat{F}_{[abc}, \hat{F}_{de]} \} , \tag{25}$$

where $\{ , \}$ denotes an anticommutator. It is easy to check that the action (23) for $p = 2$, with $\hat{F}(5)$ given by (25), is proportional to the GYM action (3) for $p = 2$.

It is clear that for general p, (25) must be replaced by

$$\hat{F}_{a_1 \cdots a_{2p+1}} = \{ \hat{F}_{[a_1 \cdots a_{p-1}}, \hat{F}_{2p, 2p+1]} \} . \tag{25'}$$

hence this procedure generalises.

Before proceeding to apply the variational principle to (23), we note an interesting property of these field strengths $\hat{F}(2p+1)$ on R^{4p+1} defined by (25'). The dual of $\hat{F}(2p+1)$ is itself a totally antisymmetric field strength of rank 2p. It is therefore plausible that the 2p-forms $F(2p)$, defined on R^{4p} and used in (2) and (3), can be related to those dual field strengths

$$*\left(\hat{F}(2p+1)\right) \stackrel{def}{=} *\hat{F}(2p) . \tag{26}$$

This turns out to be true, and as an example we consider the $p = 2$ case in detail.

Define

$$*\hat{F}_{abcd} = \frac{1}{5!} \varepsilon_{abcdefghi} \hat{F}_{efghi} . \tag{27}$$

Then it follows that *F(4), which is the 2p-form dual to

$$F(4) = F_{\mu\nu\rho\sigma} = \{ F_{\mu[\nu}, F_{\rho\sigma]} \}, \; \text{cycl. perm.} \; \nu,\rho,\sigma,$$

defined on R^8, are related to $*\hat{F}(2p)$ as follows

$$*F_{\mu\nu\rho\sigma} = 2\left(\frac{2}{1+x^2}\right)^4 \left(\hat{F}_{\mu\nu\rho\sigma} - x_\mu \hat{F}_{4\nu\rho\sigma} + x_\nu \hat{F}_{4\rho\sigma\mu} - x_\rho \hat{F}_{4\sigma\mu\nu} + x_\sigma \hat{F}_{4\mu\nu\rho}\right). \tag{28}$$

This relation, (28), is analogous to a relation noted by Adler[8] for the Abelian gauge system on R^4:

$$*F_{\mu\nu} = 2\left(\frac{2}{1+x^2}\right)^2 \left(\hat{F}_{\mu\nu} - x_\mu \hat{F}_{5\nu} + x_\nu \hat{F}_{5\mu}\right) , \tag{28'}$$

which also holds for the $p = 1$ non-Abelian case[6]. (28') and (28) are the $p = 1$ and $p = 2$ members of a hierarchy of such relations pertaining to the GYM systems in 4p-dimensions and their projections on $S^{4p} \{r|r.r = 1\}$ R^{4p+1}.

We now subject the action (23) for the special case $p = 2$ to the variational principle. The resulting Euler-Lagrange equations are

$$D_a \{ \hat{F}_{abcdi}, \hat{F}_{bcd} \} + r_i D_a \{ \hat{F}_{abcdi}, \hat{F}_{cd} \} = \delta_{ee} r_a \{ \hat{F}_{abcdi}, \hat{F}_{bcd} \} , \tag{29}$$

where D_a are the conformal covariant derivatives $D_a = r_b L_{ba} + [\hat{A}_a, \cdot]$.

Finally, the $p = 2$ BPST instanton and anti-instanton field configuration (13) is given by the following most general spherically symmetric gauge connection on R^9

$$\hat{A}_a = \alpha \; \Gamma_{ab} \; r_b . \tag{30}$$

Notice that α, which is a function of $r_a r_a = 1$, is a constant. Substituting $\hat{A}_a$ from (30) into the Euler-Lagrange equation (29) we find

$$8(\delta_{ee} - 5)(\alpha^2 - 2\alpha)^3 (\alpha - 1) \; r_b \{ \Gamma_{abcd}, \Gamma_{cd} \} = 0 \tag{31}$$

$$\Gamma_{abcd} = \{ \Gamma_{a[b}, \Gamma_{cd]} \} , \; \text{cycl. perm.} \; b,c,d.$$

It is remarkable that (31), pertaining to the case $p = 2$, has the solutions $\alpha = 0,2$ (three times), $\alpha = 1$ (once) which, apart from the degeneracy for the $\alpha = 0,2$ solutions, is the same as for the $p = 1$ case[6]. In fact for arbitrary p there is always the solution $\alpha = 1$, along with the degenerate solutions $\alpha = 0,2$ $((2p-1)$ times). The latter (degenerate) solutions lead to pure-gauge connections, thus we have the unique solution

$$\hat{A}_a = \Gamma_{ab} \, r_b \, , \tag{32}$$

for all p. To extract from (32), the $SO(4p)$ valued gauge connections pertaining to (13) we follow ref.[6] by performing the gauge transformation

$$\mathcal{U} = exp \, i \, f(r_{4p+1}) \, \Gamma_{\mu 5} \, r^\mu \, . \tag{33}$$

This removes the $\Gamma_{\mu 5}$ components of $\hat{A}_a$ in (32), thus leaving $\hat{A}_a$ only $0(4p)$-algebra components. Then A_μ, the connections we seek, can be extracted from $\hat{A}_a$ using the definitions (17). For the following choices of the functions $f(r_{4p+1})$ in (33)

$$f(r_{4p+1}) = \frac{cos^{-1} r_{4p+1}}{(1 - r_{4p+1}^2)^{1/2}} \tag{34a}$$

$$f(r_{4p+1}) = \frac{cos^{-1} r_{4p+1} - \pi}{(1 - r_{4p+1}^2)^{1/2}} \, , \tag{34b}$$

we find, for the required 4p-dimensional BPST instanton and anti-instanton connections, the two gauge equivalent expressions

$$A^\mu(x) = \frac{2}{1+x^2} \, \Gamma^{\mu\nu} x_\nu \, , \tag{35a}$$

$$A^\mu(x) = \frac{2}{(1+x^2)x^2} \, \Gamma^{\mu\nu} x_\nu \, . \tag{35b}$$

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