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8

FUNCTIONAL ANALYTIC CONTINUATION TECHNIQUES WITH APPLICATIONS IN FIELD THEORY

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Abstract.

Often one has data at points inside the holomorphy domain of a Green's function, or of an Amplitude or Form-Factor, and wants to obtain information about the spectral function i.e. the discontinuity along the cuts. Data may be experimental or theoretical. In QCD for example the perturbation expansion is valid only for unphysical values of the energy: one would like to continue this information to the cuts to find the resonance parameters. However, analytic continuation off open contours is extremely unstable. Also, the straightforward continuation of the truncated perturbation expansion will not do, since this is itself analytic and continuation will thus yield exactly the same result.

This problem is solved by functional techniques, first by allowing small imprecisions in the data to remove the uniqueness of the continuation, and then by introducing a stabilizing condition suited to the particular physical problem, which will suppress the functions with incorrect behaviour. The stabilizing condition is expressed in terms of a norm giving a measure of the smoothness of the Discrepancy Function – which is the Amplitude with the resonances removed. The minimal norm computed from the data depends on the trial values of the resonance parameters and enables one to select the best values for these. The corresponding optimal amplitude is also constructed.

An explicit solution is obtained for the case of a discrete data set; in the continuous case the problem is expressed in terms of a Fredholm integral equation.

1 Statement of the Problem

Analytic continuation is one of the main tools by means of which prediction can be achieved in Physics. Many functions of physical interest have been proven (or assumed) to be holomorphic in some domain of their relevant parameters - energy, momentum transfer and so on. Analytic continuation thus becomes an ideal conveyor of information from those regions where data is available to those of physical interest; for instance up to the region of the cuts of the Scattering Amplitude, in order to find the parameters of the resonances.

Data may not only be experimental. Analytic continuation is also a valuable tool in purely theoretical problems, for example in Field Theory in order to convey the information provided by some specific procedure (eg. perturbative expansion) far outside its natural domain of validity. There has been a strong revival of interest in these questions in the last few years, especially in asymptotically free field theories (in Q.C.D.) where the perturbative series are expected to yield sensible results only for far, nonphysical values of the energy.

The standard ways [1,3] used so far [4] to tackle these problems are based on some specially weighted contour integrals [1], on Borel summations [2] — which to some extent (see [2]) are equivalent to the former method — or on moment analysis [3]. Borel summations, for instance, make use of the information contained in the numerical value of the coefficients of the initial series, not in the form of their sum (which may diverge), but in that of a suitable integral representation which, provided some conditions are fulfilled, has a much larger domain of validity. Unfortunately, as often happens when one tries to apply existing mathematical procedures to Physics, it is very hard if not impossible to prove that these necessary conditions really hold in actual situations. The classical counter-example given by Khuri in [5] is a serious warning in this respect.

In order to understand the scope of this paper, it is important to realize that this problem *is not* simply the matter of a straightforward analytic continuation. Indeed, since the truncated perturbation function is itself an analytic function of the energy, due to the uniqueness of analytic continuation a straightforward continuation would give exactly the same perturbative function, which is known to yield false results in the resonance region. Bearing this in mind, we shall proceed as follows:

a) we shall first remove the uniqueness of the continuation by allowing (small) imprecisions into the initial data. This is legitimate, since even where the perturbative series converges well, the truncated perturbative expression is still not exact;
b) we then introduce a functional filter in the continuation procedure in order to "sieve out" any function of unwanted behaviour, as the truncated perturbative series itself is. Since unsuitable expansions behave badly especially around singular points (branching points, second Riemann sheet poles, etc.) one is able to investigate theoretically what kind of behaviour, for example at threshold and infinity, one would have to rule

out. It is perhaps good to stress that such a function-filter is in fact required in a natural way by the Functional Analysis, since without it, in the presence of the least imprecision of the initial data, *any* answer is permitted by the infinite instability of the analytic continuation process (analytic continuation off open contours is an ill-posed problem, in the Hadamard sense). The mathematical effect of such a functional filter is known to change the initial ball-topology of the infinite dimensional function space into one using neighbourhoods progressively flattened along the higher dimensions.

This filter is introduced in the form of a suitably defined norm (see eq. (2) below), the strictly positive weight $\sigma(s)$ being chosen so as to enahnce the regions where one would like to get predictions, and *to diverge* if integrated with any function of unwanted singular behaviour. Of course this norm will depend in an essential way on the kind of physical information one chooses to stabilize the continuation process. In what follows we shall use the hypothesis of the separability of the effect of the second Riemann sheet poles, i.e. of the resonances (where it holds). To this end we shall subtract from the as yet unknown physical function $A(s)$, a test function $T_\kappa(s)$ containing the resonances and any other desired features for $A(s)$; $T_\kappa(s)$ depends on some as yet undetermined parameters κ. We introduce then the Discrepancy Function [6]:

$$D_\kappa(s) = A(s) - T_\kappa(s) \qquad (1)$$

and we define a norm related to the smoothness of its imaginary part, on the cuts:

$$\delta[D\,] \equiv \{\int_{cuts}\left|\frac{\partial\,\mathrm{Im}\,D_\kappa(s')}{\partial s'}\right|^2\sigma(s')ds'\}^{\frac{1}{2}} \qquad (2)$$

It is clear that if $T_\kappa(s)$ describes correctly the structure on the cuts of A(s), $\delta[D_\kappa]$ will have a pronounced minimum for that $\kappa = \kappa_0$ which corresponds to the true positions of the resonances poles.

Of course, since on the cuts $D_\kappa(s)$ is at this stage as unknown as A(s) itself, $\delta[D_\kappa]$ cannot be computed directly. The only available information is the data $a(s_i)$ and $d_\kappa(s_i)$

$$a(s_i) \equiv A(s_i) + \varepsilon(s_i)$$

$$d_\kappa(s_i) = a(s_i) - T_\kappa(s_i) \qquad (3)$$

given at the points $s_i \in \gamma$ inside the holomorphy domain of A(s). The deviations $\varepsilon(s_i)$ are unknown, but it is supposed that they are subjected to the χ^2 condition

$$\chi^2[D_\kappa] \equiv \sum_{s_i\varepsilon\gamma}(D_\kappa(s_i)-d_\kappa(s_i))^2 n(s_i) \equiv \sum_{s_i\varepsilon\gamma}\varepsilon^2(s_i)n(s_i) \leq 1 \qquad (4)$$

where $n(s_i)$ is a given function.

Functional Analysis allows us to calculate effectively the smallest of these norms $\delta[D_\kappa]$ — call it $\delta_0(\kappa)$ — which is still compatible with the data and the χ^2-condition, *and* with the analyticity of $D_\kappa(s)$. This $\delta_0(\kappa)$ depends solely on the data $a(s_i)$ given on γ, the test function $T_\kappa(s)$ and on the weights $n(s_i)$ and $\sigma(s)$. The position $\kappa = \kappa_0$ of its minimum corresponds to the smoothest possible discrepancy function $D^0_\kappa(s)$ and hence, particularly if the minimal is a sharp one, it is a reasonable surmise that κ_0 represent the true resonance parameters. On the other hand if $\delta_0(\kappa)$ does not have any pronounced minima this result may still be significant since if much structure is still left in the discrepancy $D_\kappa(s)$ whatever values are given to κ, then the hypothesis $T_\kappa(s)$ has been shown to be wrong. This demonstrates clearly enough the interest of the δ_0 approach [7]. Moreover, if this minimum does exist, one can construct explicitly the extremal (most smooth) function $D^0_{\kappa_0}(s)$ whose norm $\delta[D^0_{\kappa_0}]$ (eq. (2)) equals $\delta_0(\kappa_0)$, and hence, via eq. (1), the optimal function $A^0(s)$ is obtained.

2. The Continuous Data Case

From now on we shall use the variable $z = z(s)$ which maps the cut complex s-plane onto the unit disk $|z| \leq 1$. The weights $n(s)$, $\sigma(s)$ will be changed by the mapping, but for the sake of simplicity we shall use the same symbols as before.

Now, if the set $\gamma \equiv \{z_i\}$ of points where the data $d_\kappa(z_i)$ are given becomes a continuum (a real line segment) then condition (4) reads

$$\chi^2[D_\kappa] = \int_\gamma (d_\kappa(z') - D_\kappa(z'))^2 n(z')dz' \leq 1 \qquad (5)$$

Since the function $D_\kappa(z)$ is determined by the boundary values of its tangential derivative, $\partial(\mathrm{Im}\,D_\kappa(z' = e^{i\phi}))/\partial\phi$, only up to an arbitrary constant, eq. (2) does not really represent a norm for $D_\kappa(z)$. However if one considers the space X of functions X(z) which vanish at some specific point $z = z_0$. (and which are holomorphic for $|z| < 1$ and have a tangential derivative $x_r(\phi)$ on the boundary which is L^2) then

$$\delta^2[X] \equiv \|X\| \equiv \frac{1}{2\pi} \int_0^{2\pi} x_r^2(\phi)\,\sigma(\phi)\,d\phi \equiv F[x_r] , \qquad (6)$$

where $x_r(\phi) \equiv \partial(\mathrm{Re}\, X(z'))/\partial r\big|_{|z'|=1} = \partial(\mathrm{Im}\, X(z'))/\partial\phi\big|_{|z'|=1}$, is indeed a norm for $\mathcal{X}$. Further

$$X(z) = -\frac{1}{\pi} \int_0^{2\pi} \ln\left(\frac{e^{i\phi}-z}{e^{i\phi}-z_o}\right) x_r(\phi)\,d\phi , \qquad (7)$$

where the kernel of eq. (7) is just the complex extension of the Neumann kernel $N(z_o;z,z') = -2\ \ln|(z'-z)/(z'-z_o)|$ (see the appendices of ref. [8]). Then, writing

$$D_\kappa(z) = X(z) + d_o, \qquad d_o \equiv D_\kappa(z_o) , \qquad (8)$$

our problem reduces to that of finding the extremum of the functional $F[x_r]$ under the constraint

$$H[x_r] \equiv \chi^2[D_\kappa] - 1 \equiv \int_\gamma dz\, n(z)\Big\{d_\kappa(z) - d_o - \frac{1}{2\pi}\int_0^{2\pi} N(z_o;z,e^{i\phi})\, x_r(\phi)\,d\phi\Big\}^2 - 1 = 0 \qquad (9)$$

In a more general case H would be a mapping from the normed space $\mathcal{X} = \{x_r\}$ to a normed space $\mathcal{Z}$: this would enable one to consider, besides eq. (9), also other constraint of physical interest, like the positivity of the spectral function ($\mathrm{Im}\, A \geq 0$) [9], and so on. This kind of extremal problem is solved using the generalised Lagrange multiplier technique (Liusternik), by constructing the functional

$$F^*_{z^*}[x_r] = F[x_r] + \langle H[x_r], z^*\rangle \qquad (10)$$

where z^* are elements of the space Z^*, dual to Z, and then asking that its Fréchet derivative with respect to x_r should

vanish. In our case where only the constraint (9) is taken into account, $Z^* \equiv Z \equiv \mathbb{R}^1$ and $\langle H[x_r], z^*\rangle \equiv \lambda H[x_r]$.

The Fréchet differential of $F^*[x_r]$ (more precisely its Gâteau differential, which, in our case, is the same) is

$$\delta F^*[x_r;y] \equiv \lim_{\alpha\to 0} \frac{\partial F^*[x_r+\alpha y]}{\partial\alpha} =$$

$$= \frac{1}{\pi}\int_0^{2\pi} d\phi\, y(\phi)\Big\{x_r(\phi)\,\sigma(\phi) - \lambda\Big[\int_\gamma dz\, n(z)\, N(z_o;z,e^{i\phi})\,(d_\kappa(z)-d_o) - \frac{1}{2\pi}\int_0^{2\pi} d\phi' \int_\gamma dz\, n(z)\, N(z_o;z,e^{i\phi})\, N(z_o;z,e^{i\phi'})\, x_r(\phi)\Big]\Big\}$$

The vanishing of $\delta F^*[x_r;y]$ for any $y(\phi)$ as well as the vanishing of $\partial F^*/\partial d_o$ (see eq. (8)) leaves us with the (Fredholm) integral equation for the boundary tangent derivative function $x_r(\phi)$:

$$(x_r\sigma^{\frac{1}{2}})(\phi) = \lambda G(\phi) + \lambda\frac{1}{2\pi}\int_0^{2\pi} d\phi'\, K(\phi,\phi')\,(x_r\sigma^{\frac{1}{2}})(\phi') \qquad (12a)$$

where, defining $n_\gamma \equiv \int_\gamma dz\, n(z)$,

$$G(\phi) \equiv \frac{1}{\sigma^{\frac{1}{2}}(\phi)}\int_\gamma dz\, n(z)\, N(z_o;z,e^{i\phi})\Big[d_\kappa(z) - \frac{1}{n_\gamma}\int_\gamma dz'\, n(z')\, d_\kappa(z')\Big] \qquad (12b)$$

$$K(\phi,\phi') \equiv \frac{1}{\sigma^{\frac{1}{2}}(\phi)\,\sigma^{\frac{1}{2}}(\phi')}\Big\{\frac{1}{n_\gamma}\int_\gamma dz\, n(z)\, N(z_o;z,e^{i\phi}) \cdot \int_\gamma dz'\, n(z')\, N(z_o;z',e^{i\phi'}) - \int_\gamma dz\, n(z)\, N(z_o;z,e^{i\phi})\, N(z_o;z,e^{i\phi'})\Big\} . \qquad (12c)$$

Further,

$$d_o = \frac{1}{n_\gamma}\int_\gamma dz\, n(z)\, d_\kappa(z) - \frac{1}{n_\gamma}\int_\gamma dz\, n(z)\,\frac{1}{2\pi}\int_0^{2\pi} d\phi\, N(z_o;z,e^{i\phi})\, x_r(\phi). \qquad (12d)$$

The value of the Lagrange multiplier λ can be found then by means of eq. (9). Once the optimal $x_r^o(\phi)$ is found (by solving eqs. 12), $X^o(z)$, and then $D_\kappa^o(z)$ and $A^o(z)$ can be computed respectively by means of eqs. (7), (8) and (1). Finally the value of the important functional δ_o is obtained by direct integration [10]

$$\delta_o(\kappa) \equiv \delta[D_\kappa^o] = \frac{1}{2\pi}\int_0^{2\pi} d\phi\; x_r^{o\,2}(\phi)\,\sigma(\phi) \qquad (13)$$

3. The Discrete Data Case

Often the set γ of points z_i where the data are given, is discrete. This is the usual situation when the data are measured experimentally but this case is important also in theoretical problems, especially where the relevant functions are evaluated with a computer. This discrete case is discussed at length elsewhere [11], we shall give here only a summary of the relevant results.

Let $d_i = d_\kappa(z_i)$ be the given, error-affected data and $\tilde{d}_i$ the (as yet unknown) corresponding values of the discrepancy function, $\tilde{d}_i \equiv D_\kappa(z_i)$, subjected to

$$\chi^2[\tilde{d}_i] \equiv \sum_{i=1}^{N} (\tilde{d}_i - d_i)^2 n_i \quad \leq 1 \qquad (14)$$

For simplicity take the reference point z_o (see preceding section) to be z_1, so that

$$X(z_i) \equiv D_\kappa(z_i) - D_\kappa(z_1) = \tilde{d}_i - \tilde{d}_1,\; X(z_1) = 0 \;. \qquad (15)$$

Now if $X^1(z)$ is some specific function satisfying eq. (15), and if $M(z)$ is holomorphic for $|z| < 1$ and $M(z_i) = 0$ for all z_i's, then $X^1(z) - M(z)$ is another $X(z)$ satisfying (15). Hence δ_o is

$$\delta_o = \inf_{\substack{M \\ (M(z_i)=0)}} \|X^1(z) - M(z)\| \qquad (16)$$

where $\|\cdot\|$ is defined by eq. (6). The effective computation of δ_o is much simplified by the duality theorem [11] which transposes the above problem into a supremum one over linear functionals y^*

$$\delta_o \equiv \inf_M \|X^1 - M\| = \sup\sup_{\substack{\|y^*\| \\ \langle M, y^*\rangle = 0,\ \text{for} \\ \text{all } M,\ M(z_i)=0.}} \langle X^1, y^*\rangle \qquad (17)$$

The result has the form

$$\delta_o = \Big\{ \sum_{i,j=2}^{N} (\alpha^{-1})_{ij} (\tilde{d}_i^o - \tilde{d}_1^o)(\tilde{d}_j^o - \tilde{d}_1^o) \Big\}^{1/2} \qquad (18a)$$

where

$$\alpha_{ij} = \frac{1}{2\pi}\int_0^{2\pi} N(z_1; z_i, e^{i\phi})\, N(z_1; z_j, e^{i\phi})\, \sigma^{-1}(\phi)\, d\phi \qquad (18b)$$

The optimal $\tilde{d}_i^o$ themselves are found by a Lagrange multiplier method combining the minimum condition for eq. (18a) with the constraints of eqs. (14) [12]. The explicit form of the result is given in ref. [11].

Hence, in the cases of both the continuous and discrete data, there exists an explicit method which permits us to assess the values of the parameters of the resonances (recall the discussion at the end of the first section), and to construct explicitly the Optimal Amplitude (Green functions, etc.) $A^o(s)$, up to the cuts.

References

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(7) If $\delta_o(\kappa)$ is small for large ranges of κ's, this means that the data are not accurate enough to support the hypothesis $T_\kappa(z)$. Hence the δ_o functional test is really exhaustive.

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(12) Alternatively, one can also use the Fréchet differential method of section 2 in the discrete case, again leading to eqs. (18) (see ref. (10)).