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Representation of Algebras over a Complete Discrete Valuation Ring.*

by

Manfred Dugas, Rüdiger Göbel and Brendan Goldsmith.

Fachbereich 6, Mathematik,
Universitat Essen GHS,
D43, Essen, Germany.

Department of Mathematics,
Dublin Institute of Technology,
Kevin Street, Dublin 8.
and
Dublin Institute for Advanced Studies,
Dublin 4, Ireland.

* Dedicated to Ines G, daughter of the second named author.

§ 0. Introduction.

It has long been known that torsion-free modules over a complete discrete valuation ring R have many nice properties not possessed by modules over incomplete discrete valuation rings. For example, every torsion-free indecomposable R-module has rank 1 and every countably generated torsion-free R-module is a direct sum of a divisible and a free module. In addition, such modules are determined by their endomorphism rings i.e. if G,H are torsion-free R-modules having isomorphic endomorphism rings, then G and H are isomorphic [18]. This also is a property not possessed by modules over incomplete discrete valuation rings. It might then be hoped that this class of modules is free from the known pathologies of decomposition which occur for Abelian groups and modules over an incomplete discrete valuation ring; we show in § 4 that this is not so. Following the lead given by Corner [1], [2], [4] such properties are usually established by realizing a suitable ring as an endomorphism ring. Now the rings which can occur as full endomorphism rings of modules over complete discrete valuation rings have been characterized by Liebert [14] (see also [11]), but unfortunately, such results are, of necessity, so complicated that they do not readily lend themselves to applications. Our approach has been to start with an R-algebra A and show that it is essentially(a term explained in § 3) the endomorphism algebra of a torsion-free R-module. In particular we establish the following:

Theorem 3.1. Let R be a complete discrete valuation ring and A a R-algebra. Then the following are equivalent:

(1) A is Hausdorff and torsion-free.

(2) There is a torsion-free reduced R-module G such that $E(G) = A \oplus \text{Ines } G$.

(3) There is a torsion-fre reduced R-module G with property (2) for any strong limit cardinal $|G|$ of cofinality greater than $A^{\aleph_0}$.

The similarity between the problems of realizing algebras as endomorphism algebras of R-modules and realizing rings as endomorphism rings of primary Abelian groups has already been noted(see [10] and [12]). The techniques used in establishing Theorem 3.1. are a modification and simplification of the techniques used previously by two of us [5] to extend and correct work by Shelah [16] on primary Abelian groups.

We complete this introduction by establishing some conventions and notations. R shall always be a complete discrete valuation ring with unique prime element p. If G is a R-module we shall denote the algebra of R-endomorphisms of G by E(G). A R-module A may always be topologized by taking the submodules $\{p^n A \mid n < \omega\}$ as a neighbourhood basis of zero. The resulting topology is the familiar p-adic or natural topology; topological references shall always be to this topology. Recall that the property of being Hausdorff is, for a R-algebra A, equivalent to A being reduced as a R-module. The books by Fuchs [8], [9] are standard references for the elementary terms used throughout this work. Finally we note that set-theoretic conventions and notations are established in §1.

§1. Set-Theoretic Preliminaries.

Standard concepts in set theory may be found in Jech [13] and most of the results listed below have been used by Dugas and Göbel in [5]. For the convenience of the reader we list some standard notation and review some elementary concepts.

$X \setminus Y = \{x \in X \mid x \notin Y\}$, $|X|$ = cardinality of the set X,

$\mathcal{P}_{\aleph_0}(X) = \{Y \subseteq X \mid |Y| = \aleph_0\}$, $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$.

The symbol ω shall be used to denote the first infinite ordinal. If $\{X_i \mid i \in I\}$ is a family of disjoint sets, then $\prod_{i \in I} X_i = \{f: I \to \bigcup_{i \in I} X_i \mid f(i) \in X_i \text{ for all } i \in I\}$. If $f: X \to Y$ is a map and $V \subseteq X$, then we write $f|V$ to denote the restriction of f to V. We shall say that sets A and B are almost disjoint if $A \cap B$ is a finite set.

Recall that an ordinal α is identified with the set of all ordinals $\beta < \alpha$ and a cardinal κ is identified with an ordinal α if $\kappa \neq |\beta|$ for all ordinals $\beta < \alpha$. A set C is cofinal in α if $\sup C = \alpha$; $\mathrm{cf}\alpha$ will denote the ordinal $\inf\{|X| \mid X \text{ is cofinal in } \alpha\}$. A cardinal κ is said to be regular if $\kappa = \mathrm{cf}\,\kappa$; otherwise it is singular. A cardinal κ is said to be a strong limit cardinal if $2^\lambda < \kappa$ for all cardinals $\lambda < \kappa$. All strong limit cardinals in this paper shall be singular. The successor ρ^+ of a cardinal ρ is the least cardinal $> \rho$.

Proposition 1.1. If κ is a strong limit cardinal then $2^\kappa = \kappa^{\mathrm{cf}\,\kappa}$ and if, in addition $\mathrm{cf}\,\kappa = \omega$, then $2^\kappa = \kappa^{\aleph_0}$.

Proof; This follows from Jech [13, (6.21) and (6.4)].

Proposition 1.2. If ρ is a cardinal, then there is a strong limit cardinal λ such that $\mathrm{cf}\,\lambda > \rho$.

Proof: Define inductively, for each $\alpha < \rho^+$, a sequence of cardinals $\{\lambda_\alpha\}$ by $\lambda_0 = \aleph_0$, $\lambda_{\alpha+1} = \sup\{\kappa = \lambda_\alpha, 2^\kappa, 2^{2^\kappa}, \ldots\}$ and $\lambda_\alpha = \sup\{\lambda_\nu \mid \nu < \alpha\}$ if α is a limit ordinal. Then the cardinal $\lambda = \lambda_{\rho^+}$ is a strong limit cardinal with $\mathrm{cf}\,\lambda = \rho^+ > \rho$.

Proposition 1.3. Let κ be an infinite cardinal and suppose $F, T \subseteq \mathcal{P}_{\aleph_0}(\kappa)$ satisfy

(a) If $f, g \in F$ and $|f \cap g| = \aleph_0$, then $f = g$.

(b) $|T| < |F|$

(c) $2^{\aleph_0} < |F|$.

Then there is a subset F' of F with $|F'| = |F|$ and such that $t \in T$, $f \in F'$ and $|t \cap f| = \aleph_0$ imply $t = f$.

Proof: Let $W = \{v \in F \mid \exists\, t \in T \text{ with } |t \cap v| = \aleph_0\}$. For each $t \in T$, let $K_t = \{v \in F \mid |v \cap t| = \aleph_0\}$. Clearly then $W = \bigcup_{t \in T} K_t$. Define, for each $t \in T$, a map $\sigma_t: K_t \to \mathcal{P}(t)$ by $\sigma_t(v) = v \cap t$, and note that it follows from (a) that σ_t is injective. Hence $|K_t| \leq |\mathcal{P}(t)| \leq 2^{\aleph_0}$. But then $|W| = \max\{2^{\aleph_0}, |T|\} < |F|$ by (b) and (c). Then $F' = F \setminus W$ is the required set.

Our next result is a simplified version of [5, 2.7.].

Proposition 1.4. Let $X = \prod_{n<\omega} X_n$ be the product of the infinite sets X_n, where $|X_n| \leq |X_m|$ if $n \leq m$. Then there is a subset F of X such that (i) $|F| = |X|$ and (ii) if $f, g \in F$ and $\sup\{n \mid f(n) = g(n)\} = \aleph_0$, then $f = g$.

Proof: Set $Y_n = \prod_{j=1}^{n} X_j$ and note that $|Y_n| = |X_n|$. Hence we may identify Y_n and X_n as sets. If $f \in X$ then f is a function with domain ω and hence may be regarded as a graph. Thus $f \cap Y_n$ is the initial segment of this graph up to n. Define a function $*: X \to Y = \prod_{n<\omega} Y_n$ by $f \to f^*$ where $f^*(n) = f \cap Y_n$. Let $F = \{f^* \mid f \in X\}$ and note that $F \subseteq Y$ and so, since $|Y| = |X|$, F may be identified with a subset of X. Moreover if $f, g \in F$ and $\sup\{n \mid f(n) = g(n)\} = \aleph_0$ then $f \cap Y_n = g \cap Y_n$ for $n \in \Lambda$ where Λ is an unbounded subset of ω. But $f \cap Y_n$, $g \cap Y_n$ are the initial segments of the graphs f, g respectively and so we conclude that $f = g$. Thus * is a bijection and (i) and (ii) follow.

§2. Algebraic Preliminaries.

Throughout this section we shall suppose that A is a torsion-free reduced R-module and μ is a singular strong limit cardinal of cofinality greater than $|A|^{\aleph_0}$; such cardinals exist by Proposition 1.2. Let $\overset{0}{\mu} = \{\lambda \mid \lambda < \mu, \lambda \text{ is a strong limit cardinal}, \mathrm{cf}\,\mu < \lambda \text{ and } \mathrm{cf}\,\lambda = \omega\}$ and let $B = \bigoplus_{\alpha<\mu} \alpha A$ be a free right A-module having ordinals $\alpha < \mu$ as free generators; $\widehat{B}$ denotes the completion of B in the p-adic topology. Thus an element x of $\widehat{B}$ has the form $x = \sum_{n<\omega} \alpha_n a_n$ where $\alpha_n < \mu$ and $\{a_n\}$ is a null sequence in the p-adic topology on A. Given such an element $x = \sum \alpha x_\alpha \in \widehat{B}$, we denote the support of x by $[x]$ i.e. $[x] = \{\alpha \mid \alpha < \mu, x_\alpha \neq 0\}$. Thus $[x] \in \mathcal{P}_{\aleph_0}(\mu)$. If $J \subseteq \mu$ we write

$||J||$ for sup$J$ and, if $x \in \hat{B}$, we shall use the shorthand $||x||$ to denote $||[x]||$.
If $I \subseteq \mu$, let $\hat{B}_I = (\bigoplus_{\alpha \in I} \alpha A)^\wedge$; $\hat{B}_I$ is canonically a direct summand of $\hat{B}$.

Definition: If $\lambda \in \mu_0$ and $X \subseteq \lambda$, we say that X is λ-big(in λ) if there is a sequence of cardinals $\lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots$ with $\lambda = \sup_{n<\omega}\{\lambda_n\}$ and $\lambda_n^+ = |X \cap \lambda_n^+|$ ($= |X \cap (\lambda_n^+ \setminus \lambda_{n-1}^+)|$.)

Proposition 2.1. Let $h \in E(\hat{B})$ and suppose $I \subsetneq \mu_0$ with $|I| \leq |A| \leq |A|^{\aleph_0} < \mathrm{cf}\,\mu$. Then there exists $\lambda \in \mu$ and a subset I^* of λ containing I, which is λ-big and $h(\hat{B}_{I^*}) \subseteq \hat{B}_\lambda$.

Proof: Since $|h(\hat{B}_I)| \leq |\hat{B}_I| \leq |\mathcal{P}_{\aleph_0}(I)||A|^{\aleph_0} \leq |I|^{\aleph_0}|A|^{\aleph_0} = |A|^{\aleph_0} < \mathrm{cf}\mu$, it follows that there is a $\lambda_0 \in \mu_0$ such that $h(\hat{B}_I) \subseteq \hat{B}_{\lambda_0}$. Now choose $\lambda_1 \in \mu$ with $\lambda_1 > \lambda_0$. Now consider $\sup\{\,||y||\ |y \in h(\hat{B}_{\lambda_1^+ \setminus \lambda_0^+})\}$. If this supremum is not μ set $I_1 = \lambda_1^+ \setminus \lambda_0^+$ and pick λ_2 such that $h(\hat{B}_{\lambda_1^+ \setminus \lambda_0^+}) \subseteq \hat{B}_{\lambda_2}$. Suppose however that $\sup\{||y||\ |y \in h(\hat{B}_{\lambda_1^+ \setminus \lambda_0^+})\} = \mu$. Let $\mu_\alpha (\alpha < \mathrm{cf}\mu)$ be an increasing chain of cardinals with $\mu = \sup \mu_\alpha$. Set $W_\alpha = \{i \in \lambda_1^+ \setminus \lambda_0^+\ |h(\hat{B}_{\{i\}}) \subseteq \hat{B}_{\mu_\alpha}\}$. Note that $h(\hat{B}_{\{i\}})$ is always contained in some $\hat{B}_{\mu_\alpha}$ since $|\hat{B}_{\{i\}}| = |\hat{A}| \leq |A|^{\aleph_0} < \mathrm{cf}\,\mu$. Next observe that $|W_\alpha| = \lambda_1^+$ for some α since if $|W_\alpha| < \lambda_1^+$ for all $\alpha < \mathrm{cf}\mu$, then we have $\lambda_1^+ = |\lambda_1^+ \setminus \lambda_0^+| = |\bigcup_{\alpha<\mathrm{cf}\mu} W_\alpha| \leq \mathrm{cf}\,\mu.\lambda_1 = \lambda_1$ ——— contradiction. Thus we have a $\alpha_1 < \mathrm{cf}\,\mu$ such that $|W_{\alpha_1}| = \lambda_1^+$. Since h is continuous we conclude that $h(\hat{B}_{W_{\alpha_1}}) \subseteq \hat{B}_{\mu_{\alpha_1}}$. Let $I_1 = W_{\alpha_1}$ and choose $\lambda_2 \in \mu_0$ such that $\mu_{\alpha_1}, \lambda_1 < \lambda_2$. Continue by induction to obtain a sequence $\lambda_0 < \lambda_1 < \ldots, \lambda_i \in \mu$, $I_n \subseteq \lambda_n^+ \setminus \lambda_{n-1}^+$, $|I_n| = \lambda_n^+$ and such that $h(\hat{B}_{I_n}) \subseteq \hat{B}_{\lambda_{n+1}}$. If $\lambda = \sup_{n<\omega} \lambda_n$, then clearly $I^* = \bigcup I_n$ is λ-big and $h(\hat{B}_{I^*}) \subseteq \hat{B}_\lambda$.

Let $\{(h_\alpha^\lambda, P_\alpha^\lambda)|\ \alpha < \kappa\}$ denote the set of all pairs of homomorphisms $h_\alpha^\lambda: P_\alpha^\lambda \to \hat{B}_\lambda$ and canonical direct summands P_α^λ of $\hat{B}_\lambda$ where $P_\alpha^\lambda = \hat{B}_{I(\lambda,\alpha)}$ and $I(\lambda,\alpha)$ is a λ-big subset of $\lambda \in \mu_0$ and $|I(\lambda,\alpha)| < \mu$. Clearly there are at most 2^λ such subsets $I(\lambda,\alpha)$. However since $\lambda^{\aleph_0} = 2^\lambda$ (by Proposition 1.1.) it follows that there are precisely 2^λ such summands P_α^λ. (cf. Dugas and Göbel [5] or Goldsmith [12].) By using many repetitions, arrange the list of all such pairs in such a way that, given an ordinal $\lambda_0 < \mu$ and a pair (h,P), there will always be a $\lambda_0 < \nu < \mu_0$ and $\rho < 2^\nu$ with $(h,P) = (h_\rho^\nu, P_\rho^\nu)$. Let $\Delta = \{(\lambda,\alpha)\ |\ \lambda \in \mu_0\ \ \alpha < 2^\lambda\}$ be the list of all such pairs ordered lexicographically i.e. $(\lambda,\alpha) < (\kappa,\beta)$ if $\lambda < \kappa$ or if $\lambda = \kappa$ and $\alpha < \beta$.

Definition: An element $x = \Sigma\, \alpha x_\alpha$ in $\hat{B}$ is said to be λ-high (for an ordinal $\lambda < \mu$) if (a) $[x] \subseteq \lambda$ (b) $||x|| = \lambda$ (c) there exists $\lambda_0 < \lambda$ such that x_α is a power of p for all $\alpha > \lambda_0$, $\alpha \in [x]$.

We now show how to construct certain elements c_α^λ, d_α^λ of P_α^λ. The construction is by transfinite induction on Δ. So suppose such elements have been constructed for $(\kappa,\beta) < (\lambda,\alpha)$. We say that c_α^λ is rigid (at stage (λ,α)) if

(i) c_α^λ is λ-high (ii) $||[c_\alpha^\lambda] \cap [c_\beta^\kappa]|| < \lambda$ for $(\kappa,\beta) < (\lambda,\alpha)$

(iii) $||[c_\alpha^\lambda] \cap [d_\beta^\kappa]|| < \lambda$ for $(\kappa,\beta) < (\lambda,\alpha)$.

To complete the construction of the elements c_α^λ, d_α^λ proceed as follows:-

I. If there is a rigid element c in P_α^λ such that $h_\alpha^\lambda(c) \notin \langle\langle B \cup \{c_\beta^\kappa A|\ (\kappa,\beta) < (\lambda,\alpha)\} + cA\rangle_*$, then choose $c_\alpha^\lambda = c$, $d_\alpha^\lambda = h_\alpha^\lambda(c)$.

II. If such a rigid element does not exist, let c_α^λ be any rigid element in P_α^λ and set $d_\alpha^\lambda = 0$.

The existence of a rigid element as required in II above is established in Lemma2.2. Before giving this result let us agree to denote by Γ the subset of Δ which corresponds to selection of elements by alternative I.

Lemma 2.2. For each (λ,α) in Δ there is a rigid element c_α^λ in $P_\alpha^\lambda = \hat{B}_{I(\lambda,\alpha)}$.

Proof: Clearly we may restrict attention to $(\lambda,\alpha) \in \Delta \setminus \Gamma$. Now since $I = I(\lambda,\alpha)$ is λ-big there is a sequence of cardinals $\lambda_0 < \lambda_1 < \ldots$ with $\sup\lambda_n = \lambda$ and $|I \cap \lambda_n^+ \setminus \lambda_{n-1}^+| = \lambda_n^+$. Set $I_n = I \cap \lambda_n^+ \setminus \lambda_{n-1}^+$ and note that $|\prod_{n<\omega} I_n| = (\sup|I_n|)^{\aleph_0} = \lambda^{\aleph_0} = 2^\lambda$ (see Jech [13, 6.4.]). By Proposition 1.4. there is a subset $F \subseteq \prod I_n$ such that $|F| = |\Pi\, I_n|$ and if $f,g \in F$ then $|f \cap g| = \aleph_0$ implies $f = g$. Identifying $f \in F$ with its image $\{f(0), f(1), \ldots\}$ we may regard F as a subset of $\mathcal{P}_{\aleph_0}(\lambda)$. Now suppose the elements c_β^κ, d_β^κ ($(\kappa,\beta) < (\lambda,\alpha)$) have been constructed. Set $T = \{[c_\beta^\kappa] \cup [d_\beta^\kappa]|\ (\kappa,\beta) < (\lambda,\alpha)\}$. Then $F, T \subseteq \mathcal{P}_{\aleph_0}(\lambda)$, $2^{\aleph_0} < |F| = 2^\lambda$, $|T| < 2^\lambda$. By Proposition 1.3. there is an element f in F which is almost disjoint from all the elements in T. If $f = \{\alpha_i\}$ and we set $c = \Sigma\, \alpha_i p^i$, then c is the required rigid element.

§3. The Main Result.

If B is a free R-module with p-adic completion $\hat{B}$ and U is any dense submodule of $\hat{B}$, then every homomorphism $f:U \to \hat{B}$ has a unique extension $\hat{f}:\hat{B} \to \hat{B}$. By abuse of notation we shall often refer to such an extension as f also. If G is a Hausdorff torsion-free R-module, then an endomorphism f of G is said to be inessential if $f(\hat{G}) \subseteq G$. It is easily verified that the set Ines $G = \{f \in E(G)|\ f$ is inessential$\}$ is an ideal of $E(G)$, the algebra of all R-endomorphisms of G. This notion is related to a concept introduced by A.L.S. Corner at the Montpellier Symposium 1967, [3], and has been used previously by one of us in the current context [12]. Further properties of Ines G may be found in [12].

Recall that a ring S is the split extension of a ring A by an ideal I of S if there exist ring homomorphisms f, g $A \overset{f}{\to} S \overset{g}{\to} A$ whose composite gf is the identity map on A and $I = \mathrm{Ker}\, g$. Such a split extension will be denoted by $S = A \oplus I$, but it should be kept in mind that the direct decomposition relates only to the additive structure of S.

We can now state the main result of this paper.

Theorem 3.1. Let R be a complete discrete valuation ring and A a R-algebra. Then the following are equivalent:

(1) A is Hausdorff and torsion-free.

(2) There is a torsion-free reduced R-module G such that $E(G) = A \oplus \mathrm{Ines}\, G$.

(3) There is a torsion-free reduced R-module G with property (2) for any singular strong limit cardinal $|G|$ of cofinality greater than $A^{\aleph_0}$.

Remark: The collection of cardinals referred to in (3) above is a class and not a set.

Proof: (3) implies (2) trivially. To see that (2) implies (1) note that G torsion-free implies $E(G)$, and hence A, is torsion-free. If G is reduced and $\phi \in \bigcap_{r \in R} rA$, then, for arbitrary g in G, $\phi(g) \in \bigcap rG = 0$. Thus $\phi = 0$ and A is Hausdorff.
The remainder of the proof is devoted to showing that (1) implies (3). So suppose that A is a torsion-free Hausdorff R-algebra and pick any singular strong limit cardinal μ of cofinality $> |A|^{\aleph_0}$. Then following the notation of §2, let $\overset{0}{\mu} = \{\lambda \mid \lambda < \mu, \lambda > \mathrm{cf}\, \mu, \lambda$ is a strong limit cardinal, $\mathrm{cf}\, \lambda = \omega\}$, $B = \bigoplus_{\alpha<\mu} \alpha A$, and for each (λ, α) in Δ let c_α^λ denote a rigid element of P_α^λ as constructed previously. We claim $G = \langle B \cup \{c_\alpha^\lambda A \mid (\lambda,\alpha) \in \Delta\}\rangle_* \subseteq \hat{B}$ is the required R-module. Since G is an A-submodule of $\hat{B}$ we can identify multiplications in G by a in A with the induced R-homomorphism and so embed A in $E(G)$.
We shall have need of the following two lemmas.

Lemma 3.2. $A + \mathrm{Ines}\, G$ is a pure subring of $E(\hat{B})$.

Proof: Suppose that there is an endomorphism $h \in E(\hat{B}) \setminus A + \mathrm{Ines}\, G$ such that $p^n h \in A + \mathrm{Ines}\, G$ for some $n < \omega$. Choose n minimal and let a in A be such that $p^n h + a$ belongs to Ines G. Since Ines G is clearly pure in $E(\hat{B})$, we have $p \nmid a$. Let $h' = p^n h + a$. By induction choose $\lambda_n \in \overset{0}{\mu}$ such that (i) $\lambda_n < \lambda_{n+1}$ (ii) $\lambda_n > \sup \bigcup_{\ell=1}^{n-1} [h'(\lambda_\ell)]$.

Let $\lambda = \sup_{n<\omega} \lambda_n$ and note that $\lambda \in \overset{0}{\mu}$. If Λ is an infinite subset of $\omega \setminus \{0, 1, \ldots n\}$ we set $x_\Lambda = \sum_{\ell \in \Lambda} \lambda_\ell p^\ell \in \hat{B}$. (For $y \in \hat{B}$ we introduce the notation $y \restriction \lambda_k$ which means the λ_k-th co-ordinate of the element y.) Now by the choice of the λ_n we have

$$h'(\lambda_\ell) \restriction \lambda_k \equiv \begin{cases} 0 & \text{if } \ell \neq k \\ a & \text{if } \ell = k \end{cases} \mod p^n .$$

Moreover $h'(\lambda_\ell) \restriction \lambda_k = 0$ if $k > \ell$. Since h' is continuous we have

$$h'(x_\Lambda) \restriction \lambda_k = \sum_{\ell \in \Lambda} (h'(\lambda_\ell) \restriction \lambda_k) p^\ell = \sum_{k \le \ell \in \Lambda} (h'(\lambda_\ell) \restriction \lambda_k) p^\ell$$

$$= \begin{cases} p^k a + \sum_{k<\ell\in\Lambda} (h'(\lambda_\ell) \restriction \lambda_k) p^\ell & \text{if } k \in \Lambda \\ \sum_{k<\ell\in\Lambda} (h'(\lambda_\ell) \restriction \lambda_k) p^\ell & \text{if } k \notin \Lambda . \end{cases}$$

Hence we conclude that

$$h'(x_\Lambda) \restriction \lambda_k \equiv \begin{cases} p^k a & \text{if } k \in \Lambda \\ 0 & \text{if } k \notin \Lambda \end{cases} \mod p^{k+n+1} \qquad \text{———(1)}$$

and so $\lambda_k \in [h'(x_\Lambda)]$ for all $k \in \Lambda$.

Set $\overset{0}{x} = \sum_{\ell<\omega} \lambda_\ell p^\ell$ and observe that, since $h' \in \mathrm{Ines}\, G$, there is an integer $s < \omega$ and a representation

$$h'(\overset{0}{x}) p^s \equiv \sum_{i=1}^{m_0} c_{\overset{0}{\alpha}_i}^{\overset{0}{\lambda}_i} \overset{0}{r}_i \mod B . \qquad \text{———(2)}$$

By the construction of λ, $h'(\overset{0}{x}) \in \hat{B}_\lambda$ and so $\overset{0}{\lambda}_i \leq \lambda$ for all $1 \leq i \leq m_0$. Let $c_{\overset{0}{\alpha}_1}^{\overset{0}{\lambda}_1} \overset{0}{r}_1$ be the summand on the right-hand side of (2) which contains infinitely many of the λ_n in its support. Then $\overset{0}{\lambda}_1 = \lambda$. Moreover the properties of the supports of the rigid elements c_α^λ imply that there exists $n_0 < \omega$ such that the sets $[c_{\overset{0}{\alpha}_i}^{\overset{0}{\lambda}_i} r_i] \setminus \lambda_{n_0}$ $(1 \leq i \leq m_0)$ are pairwise disjoint(and may be empty). Choosing a subset of the λ_n and changing numeration if necessary, we may assume that all the λ_n occur only in the support of the element $c_{\overset{0}{\alpha}_1}^{\overset{0}{\lambda}_1} = c_{\overset{0}{\alpha}_1}^{\lambda}$. Let ℓ_k be the p-height of $c_{\overset{0}{\alpha}_1}^{\lambda} \restriction \lambda_k$ and t_0 the p-height of $\overset{0}{r}_1$. Then (1) and (2) imply

$$s + k + n + 1 > \ell_k + t_0 \text{ for all } k < \omega . \qquad \text{———(3)}$$

Now choose a partition $\omega = \Lambda \cup \Lambda'$ where Λ, Λ' are two infinite disjoint subsets and set $x' = x_\Lambda$. Again $h'(x_\Lambda) \in G$ and so there is $s' < \omega$ and a representation

$$h'(x')p^{s'} \equiv \sum_{i=1}^{m'} c_{\alpha_i'}^{\lambda_i'} r_i' \mod B \qquad (4)$$

There exists an index i (and we may suppose it is $i = 1$) such that $\lambda_k \in [c_{\alpha_1}^{\lambda_1'} r_1']$ for infinitely many $k \in \Lambda$. This implies that $\lambda_1' = \lambda$ and $\alpha_1' = \overset{0}{\alpha_1}$ (= α say).

As before we can assume that all the λ_k occur in $[c_\alpha^\lambda]$ and in no other summand of the right-hand side of (4). From (1) and (4) we conclude that

$$s' + k + n + 1 > \ell_k + t_1 \text{ where } k \in \Lambda \text{ and } t_1 \text{ is the p-height of } r_1' \qquad (5)$$

and $\qquad s' + k + n + 1 \leqslant \ell_k + t_1$ if $k \in \Lambda'$. $\qquad$ (6)

Now from (3) and (6) we get

$$k + (s - t_0 + n + 1) > \ell_k \geqslant k + (s' + n + 1 - t_1) \quad \text{for all } k \in \Lambda'.$$

Since $(s - t_0 + n + 1)$ is a constant we can find a constant $c < \omega$ and an infinite subset Λ'' of Λ' such that $\ell_k = k + c$ for all $k \in \Lambda''$. Now repeating the above argument with Λ'' replacing ω, say $\Lambda'' = \Delta' \cup \Delta''$ and setting $\overset{0}{x} = x_{\Lambda''}$, $x' = x_{\Delta'}$,

we will obtain $\quad s_1' + k + 1 + n > (k + c) + t_1'$ for $k \in \Delta'$

$$s_1' + k + 1 + n \leqslant (k + c) + t_1' \text{ for } k \in \Delta'' \qquad (7)$$

This implies $s_1' + n + 1 > c + t_1'$ and $s_1' + n + 1 \leqslant c + t_1'$ ----- a contradiction. This establishes the lemma.

<u>Lemma 3.3</u>. If $h \in E(\hat{B}) \setminus A$ + Ines G, then there is an element x^* of $\hat{B}$ such that $h(x^*) \notin \langle G + x^*A\rangle_*$.

<u>Proof</u>: Since A + Ines G is pure in $E(\hat{B})$, we know that $p^n h \notin A$ + Ines G for all $n < \omega$. Thus for each a in A, there exists b_{an} in $\hat{B}$ such that $h(b_{an})p^n - b_{an}a \notin G$. Since $|A| < \mathrm{cf}\,\mu$, there is a $\lambda_0 < \mu$ such that $\{b_{an} \mid a \in A, n < \omega\} \cup \{h(b_{an}) \mid a \in A, n < \omega\} \subseteq \hat{B}_{\lambda_0}$. Choose $\lambda > \lambda_0$, $\lambda \in \overset{0}{\mu}$ and $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$ with $\sup \lambda_n = \lambda$. Set $\overset{0}{c} = \sum_{i<\omega} \lambda_i p^i$, $c' = \sum_{i<\omega} \lambda_{2i} p^{2i}$. Now claim that for all $\alpha < 2^\lambda$ and all $n < \omega$ either $[\overset{0}{c}] \setminus \lambda_n \neq [c_\alpha^\lambda] \setminus \lambda_n$ or $[c'] \setminus \lambda_n \neq [c_\alpha^\lambda] \setminus \lambda_n$. For suppose there exist $\alpha, \beta < 2^\lambda$ and $m, n < \omega$ such that $[\overset{0}{c}] \setminus \lambda_n = [c_\alpha^\lambda] \setminus \lambda_n$ and $[c'] \setminus \lambda_m = [c_\beta^\lambda] \setminus \lambda_m$.

But then $\lambda_{2i} \in [\overset{0}{c}] \cap [c']$ and so $\lambda_{2i} \in [c_\alpha^\lambda]$ for $2i \geqslant n$ and $\lambda_{2i} \in [c_\beta^\lambda]$ for $2i \geqslant m$. Hence $\|[c_\alpha^\lambda] \cap [c_\beta^\lambda]\| = \lambda$. This implies $\alpha = \beta$, which leads to the contradiction that $[c^0]$ and $[c']$ are almost equal. Hence the claim is justified.

We may then assume without loss of generality that:

(*) For all $(\lambda,\alpha) \in \Delta$ and all $n < \omega$, we have $[c^0] \setminus \lambda_n \neq [c_\alpha^\lambda] \setminus \lambda_n$.

If $h(\overset{0}{c})p^n \notin G + \overset{0}{c}A$ for all n, then we are finished. So assume $h(\overset{0}{c})p^u + \overset{0}{c}r \in G$ for some $r \in A$, $u < \omega$. Let b_{ru} be the element whose existence was established at the beginning of the proof. If $h(\overset{0}{c} + b_{ru}) \notin \langle G + (\overset{0}{c} + b_{ru})A\rangle_*$ then again we are finished, so assume there is $s \in A$, $v < \omega$ such that $h(\overset{0}{c} + b_{ru})p^v - (c^0 + b_{ru})s \in G$. Then $h(c^0)p^{u+v} + c^0rp^v \in G$ and $h(c^0 + b_{ru})p^{u+v} - (c^0 + b_{ru})sp^u \in G$. Subtracting leads to $h(b_{ru})p^{u+v} - b_{ru}sp^u + c^0(rp^v - sp^u) \in G$. Note that $rp^v \neq sp^u$ for otherwise $h(b_{ru})p^u - b_{ru}r \in G$ - - - a contradiction. But then it follows from the construction of λ and c^0 that $\|h(b_{ru})p^{u+v} - b_{ru}sp^u + c^0(rp^v - sp^u)\| = \lambda$.

Let $y = h(b_{ru})p^{u+v} - b_{ru}sp^u + c^0(rp^v - sp^u) \in G$. By definition of G there exists $t = p^k$ such that $(**)$ $\quad yt = \sum_{i=1}^{n} c_{\alpha_i}^{\eta_i} r_i \mod B$ where $(\eta_1,\alpha_1) >' (\eta_2,\alpha_2) > \ldots$ and $r_1 \neq 0$ since $[yt]$ is infinite. Let $y' = \sum_{i=1}^{n} c_{\alpha_i}^{\eta_i} r_i$. Clearly $\eta_1 \geqslant \lambda$. However if $\eta_1 > \lambda$ then there is a $\lambda' \in \overset{0}{\mu}$, $\eta_1 > \lambda' \geqslant \lambda$ such that $\|[c_{\alpha_1}^{\eta_1}] \cap [c_{\alpha_i}^{\eta_i}]\| \leqslant \lambda'$. But then there exists λ'' with $\eta_1 > \lambda'' \geqslant \lambda' \geqslant \lambda$ such that

$$[y'] \setminus \lambda'' = [c_{\alpha_1}^{\eta_1} r_1] \setminus \lambda'' = [c_{\alpha_1}^{\eta_1}] \setminus \lambda'' \subsetneqq \lambda \text{ by } (**).$$

But this contradiction then forces $\eta_1 = \lambda$.

Repeating the argument one can find a $\lambda_n < \lambda$ such that $[y'] \setminus \lambda_n = [c_{\alpha_1}^\lambda] \setminus \lambda_n$. Then enlarging n if necessary, one obtains $[yt] \setminus \lambda_n = [y'] \setminus \lambda_n = [c_{\alpha_1}^\lambda] \setminus \lambda_n$. But $[yt] \setminus \lambda_n = [c^0(rp^v - sp^u)] \setminus \lambda_n = [c^0] \setminus \lambda_n$. This forces $[c^0] \setminus \lambda_n = [c_{\alpha_1}^\lambda] \setminus \lambda_n$ and this final contradiction to (*) establishes the lemma.

Continuing with the proof of Theorem 3.1., we now show that $E(G) = A + \mathrm{Ines}\, G$. Suppose that $h \in E(\widehat{B}) \setminus A + \mathrm{Ines}\, G$. By the previous lemma we conclude that there is an element x^* in $\widehat{B}$ with the property that $h(x^*) \notin \langle G + x^*A\rangle_*$. Since cf $\mu > \omega$, there is a $\lambda_0 < \mu$ such that $x^* \in \widehat{B}_{\lambda_0}$. Now from Proposition 2.1. and the definition of the list Δ (recall the repetition of the pairs in Δ), we can find $\lambda \in \overset{0}{\mu}$ with $\lambda_0 < \lambda$, and $\alpha < 2^\lambda$ such that $x^* \in \widehat{B}_{\lambda_0} \subseteq P_\alpha^\lambda$. Moreover h maps P_α^λ into $\widehat{B}_\lambda$ and $h|P_\alpha^\lambda = h_\alpha^\lambda$. We claim that this pair (λ,α) belongs to Γ. For suppose $(\lambda,\alpha) \in \Delta\setminus\Gamma$. Then

$$h_\alpha^\lambda(c_\alpha^\lambda) \in \langle\langle B \cup \{c_\beta^\kappa A \mid (\kappa,\beta) < (\lambda,\alpha)\}\rangle + c_\alpha^\lambda A\rangle_* \text{ and so } h_\alpha^\lambda(c_\alpha^\lambda)p^n - c_\alpha^\lambda a_0 \in G \qquad \text{(X)}$$

where $a_0 \in A$, $n < \omega$. Now consider the element $x^* + c_\alpha^\lambda$. Since c_α^λ is λ-high, and $\|x^*\| < \lambda_0 < \lambda$, the element $x^* + c_\alpha^\lambda$ is also λ-high. Moreover conditions (ii) and (iii) for a rigid element are satisfied by $x^* + c_\alpha^\lambda$ since the corresponding conditions are satisfied by c_α^λ. So $x^* + c_\alpha^\lambda$ is a rigid element at stage (λ,α) and since (λ,α) is assumed to belong to $\Delta\setminus\Gamma$, we can conclude that $h_\alpha^\lambda(x^* + c_\alpha^\lambda) \in \langle G + (x^* + c_\alpha^\lambda)A\rangle_*$ and so

$$h_\alpha^\lambda(x^* + c_\alpha^\lambda)p^m - (x^* + c_\alpha^\lambda)a_1 \in G \text{ for some } a_1 \text{ in } A,\ m < \omega. \qquad \text{(Y)}$$

Subtraction of p^m times (X) from p^n(Y) gives

$$h_\alpha^\lambda(x^*)p^{n+m} - x^*a_1p^n + c_\alpha^\lambda(a_0p^m - a_1p^n) \in G.$$

But $c_\alpha^\lambda(a_0p^m - a_1p^n) \in G$ and so we conclude that $h_\alpha^\lambda(x^*)p^{n+m} \in G + x^*A$, which contradicts the defining property of x^* since $x^* \in P_\alpha^\lambda$ and $h|P_\alpha^\lambda = h_\alpha^\lambda$. This shows that $(\lambda,\alpha) \in \Gamma$.

Having established that $(\lambda,\alpha) \in \Gamma$, we now show that h does not belong to $E(G)$. As before let $d_\alpha^\lambda = h_\alpha^\lambda(c_\alpha^\lambda)$ and suppose $d_\alpha^\lambda \in G$. Then we may write

$$d_\alpha^\lambda p^m \equiv \sum_{i=1}^{n} c_{\alpha_i}^{\lambda_i} a_i \bmod B, \text{ where } a_i \in A,\ m < \omega$$

and we may suppose $(\lambda_1,\alpha_1) > (\lambda_2,\alpha_2) > \ldots (\lambda_n,\alpha_n)$. From the properties of rigid elements we conclude that $(\lambda_1,\alpha_1) \leqslant (\lambda,\alpha)$ and so $d_\alpha^\lambda \in \langle\langle B \cup \{c_\beta^\kappa A \mid (\kappa,\beta) < (\lambda,\alpha)\}\rangle + c_\alpha^\lambda A\rangle_*$. However since $d_\alpha^\lambda = h_\alpha^\lambda(c_\alpha^\lambda)$, this contradicts the fact that $(\lambda,\alpha) \in \Gamma$. So we must conclude that $d_\alpha^\lambda = h_\alpha^\lambda(c_\alpha^\lambda) = h(c_\alpha^\lambda) \notin G$ and, since $c_\alpha^\lambda \in G$, we arrive at the desired result, $h \notin E(G)$. So $E(G) = A + \mathrm{Ines}\, G$.

Finally it is clear from the construction of G that $A \cap \mathrm{Ines}\, G = 0$ and so the ring $E(G)$ is the required split extension $A \oplus \mathrm{Ines}\, G$. This completes the proof of Theorem 3.1.

Remark: The basic idea in the proof of Theorem 3.1. is used by Dugas and Göbel in [7] to prove the following complementary result:-

Theorem Let R be a Dedekind domain. Then the following are equivalent:

(1) R is not a complete discrete valuation ring.

(2) There is a R-module M of rank > 1 with $E(M) \cong R$.

(3) If A is any cotorsion-free R-algebra, then there is a R-algebra N with $E(N) \cong A$.

(For further details of cotorsion-free modules see [6].)

§4. Applications.

It is, by now, standard to apply a result such as Theorem 3.1. to exhibit a variety of pathologies relating to decomposition properties in the class of modules under discussion. Many such examples have been constructed by Corner [1], [2], [4], Dugas and Göbel [5] and Warfield [17] for various classes of modules. Here we shall restrict attention to three simple examples which show that modules over a complete discrete valuation ring are as pathological as most other classes of modules, at least in relation to decomposition properties. Since the details of the constructions we use are all well known, we only give brief outlines of the proofs.

Definition: If $\mathcal{P}$ is a class of R-modules, for some ring R, we say that a R-module G is essentially $\mathcal{P}$-indecomposable if in any direct decomposition $G = A \oplus B$, one of the summands A,B belongs to $\mathcal{P}$.

Thus if $\mathcal{P}$ is the class of bounded p-groups then essentially $\mathcal{P}$-indecomposable corresponds to the notion of essentially indecomposable used by Pierce [15] and Dugas and Göbel [5].

We shall let $\mathcal{C}$ denote the class of complete modules over an arbitrary complete discrete valuation ring R.

Example 1. There exist essentially $\mathcal{C}$-indecomposable incomplete modules of arbitrary large cardinality.

Proof: Given any cardinal ρ choose a strong limit cardinal $\mu > \rho$ with cf $\mu > |R|^{\aleph_0}$. Such cardinals exist by Proposition 1.2. By Theorem 3.1. there exists a R-module G of cardinality μ with $E(G) = R \oplus$ Ines G. Clearly such a module G is not complete. If $G = A \oplus B$ with associated projections π_1 and π_2, it follows, as in [4], that since R is a domain, one of π_1, π_2 is inessential; say $\pi_1 \in$ Ines G. But then $A \subsetneq \hat{G}\pi_1 \subsetneq G\pi_1 = A$ and so A, being a homomorphic image of the complete module $\hat{G}$, is also complete. Further examples of this type may be found in Goldsmith [12].

Example 2. Given an arbitrary positive integer t, there exists a module G over a complete discrete valuation ring R such that $G^m \cong G^n$ if and only if $m \equiv n \bmod t$.

Proof: Let A be the R-algebra freely generated by the symbols u_i, v_i $(i = 1,2, . . t+1)$ subject to the relations $\sum_{i=1}^{t+1} u_i v_i = 1$, $v_i u_j = \delta_{ij}$. Then by Theorem 3.1. there is a R-module G with $E(G) = A \oplus$ Ines G. The remainder of the proof follows as in Corner [4, pp290-1].

Example 3. There exist essentially $\mathcal{C}$-indecomposable modules over a complete discrete valuation ring with properties as in Example 2.

Proof: Full details of the construction of a suitable ring A which is an integral domain, are given in Dugas and Göbel [5, Theorem4.3.]. Since the quotient E(G)/Ines G $\cong$ A is a domain, the module G constructed according to Theorem 3.1. is, as in the proof of Example 1, essentially $\mathcal{C}$-indecomposable.

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