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A note on the higher-order Noether symmetries of Sarlet and Cantrijn

by

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Short Title

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Abstract

A significant class of the higher-order Noether symmetries for a given Lagrangian L recently described by Sarlet and Cantrijn are shown to be ordinary Noether symmetries associated with equivalent Lagrangians generated by the symmetries themselves.

I have recently attempted a classification of dynamical symmetries in Lagrangian mechanics according to the induced transformation properties of the exterior derivative of the Cartan form (Prince 1982a). The results of this investigation throw some light on the so-called "higher-order Noether symmetries" of Sarlet and Cantrijn (1981a).

In these authors' notation, a dynamical symmetry of a system with Lagrangian L is a vector field, described locally by

$$Y = \tau\frac{\partial}{\partial t} + \xi^k\frac{\partial}{\partial q^k} + \eta^k\frac{\partial}{\partial \dot{q}^k} \quad , \tag{1}$$

on $\mathbb{R} \times TM$ (M is the configuration space) with the property that

$$[Y, \Gamma] = -\Gamma(\tau)\Gamma . \tag{2}$$

Here Γ is the characteristic vector field of the exterior derivative of the Cartan form

$$\theta_L = L\,dt + \frac{\partial L}{\partial \dot{q}^k}(dq^k - \dot{q}^k dt) \tag{3}$$

(see, for example, Cartan (1922) or Hermann (1968)). Upon normalization, $\langle \Gamma, dt\rangle = 1$, can be written locally as

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^k\frac{\partial}{\partial q^k} + \Lambda^k\frac{\partial}{\partial \dot{q}^k} \quad , \tag{4}$$

where the Λ^k satisfy the Euler-Lagrange equation

$$\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\Lambda^j = \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial t} - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j . \tag{5}$$

The integral curves of Γ are just the classical trajectories of the system (on M) lifted to $\mathbb{R} \times TM$.

My main result (Prince 1982a) is that a vector field Y generating a dynamical symmetry of the system has the property

$$\mathcal{L}_Y \theta_L = \theta_{L^*} + df \tag{6}$$

with

$$i_\Gamma d\theta_{L^*} = 0 \tag{7}$$

(f: $\mathbb{R} \times TM \rightarrow \mathbb{R}$) if and only if

$$S_{[ij]} = 0 \quad , \tag{8}$$

where

$$S_{ij} = \cancel{\frac{\partial L}{\partial \dot{q}^i}\frac{\partial \tau}{\partial \dot{q}^j}} + \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^k}\left(\frac{\partial \xi^k}{\partial \dot{q}^j} - \dot{q}^k\frac{\partial \tau}{\partial \dot{q}^j}\right) . \tag{9}$$

The significance of this result is that it includes the transformation properties of θ_L under the important classes of dynamical symmetries, namely the so-called "point symmetries" or Lie symmetries and the "Noether symmetries" of Sarlet and Cantrijn (1981b) (Noether and Cartan symmetries in Prince 1982a). It is also a property of all dynamical symmetries of one-dimensional systems. In particular a dynamical symmetry satisfying (9) generates an equivalent Lagrangian L^* for the system (L^* is zero for the first two classes of symmetries mentioned above and may just be a multiple of L, see Prince 1982a).

Sarlet and Cantrijn (1982a) have recently uncovered another type of dynamical symmetry which they call n^{th}- order Noether symmetries. In this letter I demonstrate that a significant class of these symmetries are just

Noether symmetries (albeit associated with another Lagrangian).

An n^{th} - order Noether symmetry Y has the property that there is a smallest natural number n for which

$$\mathcal{L}_Y^n d\theta_L = 0. \tag{10}$$

Sarlet and Cantrijn (1982a) show that this vector field Y is associated with a closed form constant of the motion F, given by

$$\mathcal{L}_Y^{n-1}(i_Y d\theta_L) = F. \tag{11}$$

Now suppose that Y also satisfies (8), then (6) can be rewritten as

$$\mathcal{L}_Y d\theta_{L_0} = d\theta_{L_1} \tag{12}$$

where L_0, L_1 replace L and L' respectively. Suppose further that

$$\mathcal{L}_Y d\theta_{L_j} = d\theta_{L_{j+1}} \tag{13}$$

and

$$i_Y d\theta_{L_{j+1}} = 0 \quad , \quad j = 0, 1, \ldots, n-2 \quad , \tag{14}$$

that is, Y "continues to satisfy" (8). (L_j is called the j^{th} Lagrangian equivalent to L_0 associated with Y, Prince 1982b). Thus by virtue of (10)

and (13)

$$\mathcal{L}_Y^n d\theta_L = \mathcal{L}_Y d\theta_{L_{n-1}} = 0 \quad , \tag{15}$$

and by the Cartan result (see, for example, Crampin 1977) this yields a constant of motion F:

$$i_Y d\theta_{L_{n-1}} = dF \quad . \tag{16}$$

Moreover, F has the expected transformation property

$$Y(F) = 0 \; . \tag{17}$$

Results (11) and (17) are equivalent since

$$i_Y d\theta_{L_{n-1}} = i_Y \mathcal{L}_Y^{n-1} d\theta_L = \mathcal{L}_Y^{n-1}(i_Y d\theta_L) \quad . \tag{18}$$

In summary, a significant class of higher order Noether symmetries for a given Lagrangian are simply ordinary Noether symmetries for equivalent Lagrangians generated by the symmetries themselves.

References

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