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RINGS OF MONOPOLES WITH DISCRETE AXIAL SYMMETRY:

EXPLICIT SOLUTION FOR N=3

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Abstract.   It is shown that, in contrast to continuous axial
symmetry, discrete axial symmetry admits separated SU(2) monopoles
in static equilibrium. The Corrigan-Goddard conditions on the
parameters are enormously simplified and for 3 equidistant
monopoles are identically satisfied.

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The Corrigan-Goddard (CG) Ansatz[1] for n separated SU(2) monopoles in static equilibrium contains $n(n-2)$ conditions for $n(n+2)$ parameters, in accordance with the index theorem. However the $n(n-2)$ conditions are such that, in general, it is not possible to solve them explicitly or to relate the independent parameters directly to the physical properties of the system. For this reason it may be of interest to simplify the conditions by imposing symmetries on the system.

The purpose of this note is to consider those symmetries that can be implemented by linear transformations of the Ward-Atiyah ζ -variable.[2] These are the reflexions of the coordinates x,y,z and the rotations φ around the z-axis. However, since the group of continuous rotations around the axis does not admit separated monopoles,[3] only the _discrete_ subgroups R_k where $\varphi = 2\pi m/k$ $m = 1 \dots k$ will be considered. It is found that for R_n and R_{n-1} the CG conditions simplify enormously and that, for R_n and small values of the parameters at least, the system described is a (non-zero) ring of monopoles with equal spacing. In particular for n=3 and R_3 the CG conditions are automatically satisfied and there is an explicit solution (more precisely an explicit transition matrix) which describes 3 equidistant monopoles. The fact that the discrete axial symmetries R_k admit separated monopoles located at the discrete angles of the group, throws some light[†] on the rather surprising earlier result[3] that continuous axial symmetry does not admit separated solutions, since the continuum limit of an R_k system would require an infinite number of monopoles on the ring and hence an infinite energy.

Let us first recall the essential features of the CG-Ansatz. Let

$$g = \begin{bmatrix} \rho & (-\zeta)^{-n} e^{-K_{n-1}} \\ \zeta^{n} e^{-K_{n-1}} & H_n e^{-K_{n-1}} \end{bmatrix} , \qquad (1)$$

be the Ward transition matrix[4] and let

$$H_n = \gamma^n + a_{n-1}\gamma^{n-1} + \cdots + a_1\gamma + a_0 = \prod_{r=1}^{n}(\gamma - \omega_r) , \quad \gamma = 2\zeta + x_+\zeta - x_-\zeta^{-1} ; \qquad (2)$$

where the coefficients $a_r(\zeta, \zeta^{-1})$ are polynomials of degree $(n-r)$ in ζ and ζ^{-1} and are hermitian in the sense that $\overline{a(\zeta, \zeta^{-1})} = a(-\bar{\zeta}^{-1}, -\bar{\zeta})$ Then the CG-Ansatz consists of choosing

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[†] Note that a previous interpretation in which the continuous axial symmetry becomes twisted as the monopoles separate is incorrect.[9]

$$\mathcal{P} = \left(e^{K_{n-1}} + (\cdot)\, e^{-K_{n-1}}\right)/H_n \text{ where } K_{n-1} = \frac{\pi i}{2}\sum_{r=1}^{h} n_r \frac{1}{S\neq r}\prod^{h}\left(\frac{\gamma-\omega_s}{\omega_r-\omega_s}\right) = \ell + \ell_1\gamma + \cdots + \ell_{n-2}\gamma^{n-1}, \quad (3)$$

and the integers n_r are such that K_{n-1} is linear in γ in the axisymmetric limit[5]. That is, $n_r = \pm(1,3,5,\ldots,n-1)$ for even n and $n_r = (0, \pm2, \pm4,\ldots,n-1)$ for odd n. The coefficients $b_r(\xi,\bar\xi')$ in (3) are not polynomials, but, as will be seen later, they inherit the symmetry properties of the $a_r(\xi,\bar\xi')$. The polynomial H_n contains the $n(n+2)$ parameters mentioned earlier, and the $n(n-2)$ conditions for them may be expressed[1] as

$$\frac{1}{2\pi i}\oint \frac{d\gamma}{\xi}\left(\frac{\partial K_{n-1}}{\partial\gamma}\right) = c \quad \text{or} \quad \frac{1}{2\pi i}\oint\frac{d\xi}{\xi}\, b_s\,\xi^m = 0, \quad \begin{array}{l} s=1\ldots n-2 \\ -s\leq m\leq s \end{array} \quad (4)$$

where c is a constant (identified as the Higgs constant). It will be convenient to treat c as a free parameter and regard the right hand equations in (4) as the CG-conditions. Note that the conditions simply state that the first (2s+1) Laurent moments of the b_s are zero.

The linear transformations of the ξ-variable mentioned above are obtained by requiring that when the coordinates x,y,z and the variable ξ are simultaneously transformed, the quantity γ in (2) remains invariant (or reverses its sign for z→ -z). It is then easy to see that (x,y,z) reflexions and φ-rotations correspond to $\xi\to\bar\xi$, $\xi\to-\bar\xi'$ and $\xi\to\xi e^{-i\varphi}$ respectively. From (1) and (2) one then sees that the system will be invariant with respect to these transformations if H_n is invariant (or reverses its sign) and that such will be the case if the coefficients $a_r(\xi,\bar\xi')$ are invariant (or reverse their signs) under the transformations of ξ alone. Thus the invariance may be expressed completely in terms of the behaviour of the $a_r(\xi,\bar\xi')$ with respect to the transformations of ξ.

Before considering discrete axial symmetries let us first consider the reflexions. From (2) and (3) it can be seen that any reflexion symmetry of the a_r is inherited by the corresponding b_r and hence we can introduce a quantity $C_r(\xi,\bar\xi')$ to denote either a_r or b_r. In terms of C_r one can then construct the following table:

Table 1.

Reflexion	x	y	z	xy parity; x,y and $\bar z$
Condition	$C_r(\xi)=C_r(\bar\xi)$	$C_r(\bar\xi)=(-)C_r(-\bar\xi)$	$C_{2r+1}(\xi)=0$	$C_{2r}(\xi)=C_{2r}(\bar\xi^2+\bar\xi^{-2})$
No. of parameters	$\frac{n}{2}(n+3)$	$\frac{n}{2}(n+3)$	$m(2m+3)$	$\frac{m}{2}(m+3)$
No. of conditions	$\frac{1}{2}(n-2)(n+1)$	$\frac{1}{2}(n-2)(n+1)$	$(m-1)(2m+1)$	$\frac{1}{2}(m-1)(m+2)$

Here and throughout m is defined so that n=2m or n=2m+1. Note that for x,y and (x and y) reflexions the hermiticity requires that the C_r be real functions of $\xi-\bar\xi'$, $i(\xi+\bar\xi')$ and $\xi^2+\bar\xi^2$ respectively.

We should point out that z-reflexion leaves the transition matrix invariant only up to a gauge transformation.

Let us next consider R_n-invariance. Since this corresponds to $\xi\to\xi e^{-i\varphi}$ for $\varphi=2\pi m/h$, $m=1,\ldots,n$ and the only powers ξ^ℓ for $\ell\leq n$ which are invariant with respect to such rotations are ξ^0 and ξ^n one sees that R_n-invariance reduces the polynomial H_n in (2) to the form

$$H_n = \gamma^n + a_{n-1}\gamma^{n-1} + \cdots + a_1\gamma + a_0 + \epsilon\xi^n + \bar\epsilon(-\xi)^n, \quad (5)$$

where all the coefficients are independent of ξ and only ϵ is complex. The n(n+2) parameters in (2) are thus reduced to (n+2). One of the a_r may be eliminated by a suitable choice of origin on the z-axis $(\gamma\to\gamma+\frac{z}{2})$ and ϵ may be made real by a suitable choice of azimuthal angle $(\xi\to\xi e^{i\varphi})$ thus reducing the (n+2) parameters to n. Note that when ϵ is real, the system is x-reflexion invariant for even n, and y-reflexion invariant for all n.

One must then consider the CG-conditions for the system (5). The point now is that since H_n in (5) is R_n-invariant, so are the a_r and the coefficients $\ell_r(\xi,\bar\xi')$ in (3). But that means that the $\ell_r(\xi,\bar\xi')$ are functions of ξ^n and $\bar\xi^{-n}$ only. In that case the CG conditions (4) are automatically satisfied for $m\neq0$ leaving only the (n-2) conditions

$$\frac{1}{2\pi i}\oint\frac{d\xi}{\xi}\,\ell_s = 0, \quad s = 1,\ldots,n-2 \quad (6)$$

Thus R_n-invariance reduces the n(n-2) conditions (4) to the (n-2) conditions for n parameters. (One of the 2 free parameters is the Higgs constant). As a matter of fact the (n-2) conditions (6) may be regarded as normalization conditions for the (n-2) constants a_r in (5) and it is easy to see that, to first order in ϵ of equation (7) below, they are satisfied by choosing the a_r to have their axisymmetric values. Thus

to first order in ϵ the Ansatz $H_n = H_n^S + \epsilon(\zeta^n + \zeta^{-n})$ where H_n^S is the axisymmetric Ansatz, actually satisfies all the CG-conditions and hence furnishes an explicit solution for all n.

The R_n-invariant Ansatz (5) is not automatically z-reflexion invariant and if we impose z-reflexion invariance it reduces further to

$$H_n = \gamma^n + a_{n-2} \gamma^{n-2} + \cdots + a_{n-2m} \gamma^{n-2m} + \epsilon \left(\zeta^n + (-\zeta)^{-n} \right) \quad , \qquad (7)$$

where n=2m or 2m+1 and the azimuthal angle has been chosen to make ϵ real. There are only m+1 parameters in (7). On the other hand, from the table 1 we see that for z-reflexion invariance the $\ell_{2r}(\zeta)$ are odd in ζ and hence the odd-order conditions in (6) drop out, leaving only the (m-1) conditions

$$\frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} \ell_{2s}(\zeta) = 0 \qquad s = 1 \cdots m-1 \quad , \qquad (8)$$

for the (m+1) parameters. (Actually, for even n the $\ell_{2r+1}(\zeta)$ are identically zero). Thus the combination of R_n and z-reflexion invariance reduces the n(n-2) conditions for n(n+2) parameters to (m-1) conditions for (m+1) parameters, where n=2m or 2m+1. In particular, for n=2 and n=3 there are no conditions.

The Ansatz (7) is hermitian, has total monopole charge n, and is regular in the neighbourhood of its axisymmetric limit if the limit is regular (which is true for low values of n and very likely for all n)[5][6]. One might ask however, what kind of configuration the Ansatz actually describes. The R_n-symmetry implies that any monopole (zero of the Higgs field $\Phi(x)$) off the z-axis must be accompanied by n-1 other monopoles all lying on a ring. Hence the Ansatz must describe either a ring of monopoles with equal spacing or a set of monopoles on the z-axis. Actually the monopoles on the z-axis would have to be at the origin since separated monopoles would be inconsistent with the fact that in the axi-symmetric limit the zero at the origin is non-degenerate in the z-direction[5][6]. (Note that for odd n the z-reflexion invariance would force at least one monopole to lie at the origin). But now a direct computation of $\Phi(o)$ for small ϵ shows that it is not zero for any n and hence the system describes a non-zero ring of monopoles. (The computation is facilated by noting that because of the R_n-symmetry we have $\Phi(o) = |\Delta_0^{-1} \partial_+ \Delta_{n-1}|$ where Δ_r are the usual moments[6][7], and that only the coefficients ℓ and ℓ_0 of K_{n-1} enter the computation. Furthermore $\ell = 0$ for even n and $\ell = -\epsilon(\zeta^n - \zeta^{-n})/a_1$ for odd n. In fact for odd n $|\Phi(o)| = \epsilon$).

As illustration let us consider n=2,3,4. For n=2 the Ansatz (7) reproduces the separated 2-monopole solution of Ward. For n=3, by suitably normalizing the coordinates we have

$$H_3 = \gamma^3 + \gamma + \epsilon(\zeta^3 - \zeta^{-3}) \quad \text{and} \quad K_2 = \frac{(\gamma - \omega_3)[(\gamma - \omega_2)(\omega_3 - \omega_2) + (\gamma - \omega_1)(\omega_3 - \omega_1)]}{(\omega_2 - \omega_1)(\omega_3 - \omega_1)(\omega_3 - \omega_2)} \quad , \qquad (9)$$

where $\omega_1 + \omega_2 + \omega_3 = 0$ Equations (4) then reduce to

$$\frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} \frac{(\omega_1^2 + \omega_2^2 - \omega_3^2)}{(\omega_1 - \omega_2)(\omega_2 - \omega_3)(\omega_3 - \omega_1)} = c \quad \text{and} \quad \frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} \frac{2\omega_3 - \omega_1 - \omega_2}{(\omega_1 - \omega_2)(\omega_2 - \omega_3)(\omega_3 - \omega_1)} = 0 \quad . \qquad (10)$$

The first equation simply determines c for the given coordinate normalization and the second is the CG-condition coming from (8). However because the integrand is odd[††] in ζ the CG-condition is automatically satisfied. Thus the Ansatz (9) constitutes an explicit solution. It obviously describes 3 equidistant monopoles.

For n=4 by suitably normalizing the coordinates, we have

$$H_4 = \gamma^4 + \gamma^2 + a + \epsilon(\zeta^4 + \zeta^{-4}) \quad , \quad K_3 = \frac{(\sigma^2 - 3\omega^3)\gamma + (\sigma - 3\omega)\gamma}{\omega \sigma (\omega^2 - \sigma^2)} \quad , \qquad (11)$$

where a, ϵ are parameters and $\pm(\omega, \sigma)$ are the roots of $H_4 = 0$ for γ. Since K_3 contains no even terms in γ, the odd coefficients ℓ_r in (3) are zero, and so the equations (4) reduce to

$$\frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} \frac{\sigma^2 - 3\omega^2}{\omega \sigma(\omega^2 - \sigma^2)} = c \quad \text{and} \quad \frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} \frac{3\omega - \sigma}{\omega \sigma(\omega^2 - \sigma^2)} = 0 \quad , \qquad (12)$$

As in the n=3 case the first equation simply normalizes c relative to the coordinates. Thus there is one CG condition. It is not automatically satisfied and relates the parameters a and ϵ. The system described is a set of 4 monopoles located at the corners of a square.

Finally let us consider R_{n-1}-symmetry (for $n \geq 3$). Since the only powers ζ^ℓ for $\ell \leq n$ which are R_{n-1}-invariant are ζ^0 and ζ^{n-1} we see that R_{n-1}-symmetry reduces the polynomial H_n in (2) to

$$H_n = \gamma^n + a_1 \gamma^{n-1} + \cdots + a_{n-2} \gamma^2 + [\alpha_1 + \beta_1 \zeta^{n-1} - \bar{\beta}_1 \zeta^{-n+1}]\gamma + \alpha_0 + \beta_0 \zeta^{n-1} - \bar{\beta}_0 \zeta^{-n+1} \quad , \qquad (13)$$

where all the parameters are independent of ζ and only the $\beta's$ are complex.

[††] To see this more explicitly note that $\omega_3 \leftrightarrow -\omega_3$ and $\omega_1 \rightleftarrows -\omega_2$ as $\zeta \to -\zeta$
For example for small ϵ, $\omega_3 = -\epsilon(\zeta^3 - \zeta^{-3})$, $\omega_{1,2} = \pm i + \omega_3/2$.

By suitably choosing the origin on the z-axis and the azimuthal orientation, one of (a_i, α_i) can be set equal to zero, and one of the β_i made real. Thus (14) contains essentially (n+2) parameters. On the other hand, because the roots ω_r, and hence the coefficients ℓ_s in (3), are R_{n-1}-invariant, the Laurant expansions of the ℓ_s are expansions in ζ^{n-1} and ζ^{1-n}. Hence the CG-conditions (4) are automatically satisfied for m≠0, and reduce again to the zero-moment conditions (7). We then have (n-2) conditions for the (n+2) parameters in (13).

The R_{n-1} symmetry implies that (13) describes a ring of n-1 monopoles, together with a single monopole on the z-axis, or else a set of n monopoles which are all located on the z-axis. The analogy with the R_n-case, suggests that it describes the ring for non-trivial values of the parameters, but we have verified this only for the colinear n=3 case. Assuming that (13) does describe a ring, the 4 free parameters could be identified as the Higgs constant, the radius of the ring, the distance between the single monopole and the ring centre, and one internal variable. For example, for n=3 we have, for a suitable choice of origin and orientation,

$$H_3 = \gamma^3 + [\alpha_1 + \beta_1(\zeta^2 + \zeta^2)]\gamma + \alpha_0 + \beta_0\zeta^2 + \bar{\beta}_0\zeta^{-2} \quad , \tag{14}$$

where only β_0 is complex. There is only one CG-condition, namely the one shown in (12), and this condition is not automatically satisfied.

In contrast to (8) the Ansatz (13) is not automatically γ-reflexion invariant, since that would require that both β's in (13) be real. The Ansatz is also not z-reflexion invariant and z-reflexion invariance reduces it to

$$H_{2m} = \gamma^{2m} + a_2\gamma^{2m-2} + \cdots + a_{2m-2}\gamma^2 + \beta(\zeta^{2m-1} - \zeta^{1-2m})\gamma + \alpha \quad , \tag{15}$$

$$H_{2m+1} = \gamma\left\{\gamma^{2m} + a_2\gamma^{2m-2} + \cdots + a_{2m-2}\gamma^2 + [\alpha + \beta(\zeta^{2m} + \zeta^{-2m})]\right\} \quad ,$$

for even and odd n respectively. In (15) the azimuthal orientation has been chosen so that the single β which occurs is real, and the system is automatically γ-reflexion invariant. There are m+1 parameters in each case in (15), and since the z-reflexion invariance means that the odd-order conditions in (9) are automatically satisfied, there are (m-1) conditions for these (m+1) parameters. Since the single monopole and the ring-centre must now coincide, the 2 free parameters are presumably the Higgs constant and the radius of the

ring. Note that for odd n the expression in (16) is just γ multiplied by the expression for even n in (7). For n=3, (15) reduces to

$$H_3 = \gamma\left[\gamma^2 + \alpha + \beta(\zeta^2 + \zeta^{-2})\right] \quad , \tag{16}$$

the CG-condition (12) is automatically satisfied and the explicit solution is the same as the colinear n=3 solution found in a different manner by Brown, Prasad and Rossi[8].

In conclusion it might be remarked that one could continue along the same lines and consider R_s-symmetry for all s≤n. However, for s≤n-2 the CG-conditions for the special moments $m = s, 2s, 3s \cdots \le n-2$ are not automatically satisfied so the system is a little more complicated.

REFERENCES

(1)   E. Corrigan, P. Goddard, Comm. Math. Phys. 80, 575 (1981).

(2)   M. Atiyah, R. Ward, Comm. Math. Phys. 55, 117 (1977)

(3)   P. Houston, L. O'Raifeartaigh, Proc. Conf. Diff. Geom. Methods, Clausthal 1980 (ed. Doebner, Springer, Berlin 1981).

(4)   R. Ward, Comm. Math. Phys. 79, 317 (1981).

(5)   M. Prasad Comm. Math. Phys. 80, 137 (1981).

(6)   L. O'Raifeartaigh, S. Rouhani Schladming 1981 Lecture Notes, Acta Physica Austriaca (in Press).

(7)   E. Corrigan, D. Fairlie, P. Goddard and R. Yates, Comm. Math. Phys. 58, 223 (1978).

(8)   S. Brown, M. Prasad and P. Rossi, M.I.T. Preprint.

(9)   L. O'Raifeartaigh and S. Rouhani, Phys. Lett. 105B, 177 (1981), Errata, 105B, 489 (1981).