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SELF DUAL SOLUTIONS TO GAUGE FIELD EQUATIONS ON N DIMENSIONAL MANIFOLDS

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Lectures given at the 'Mathematical Physics Colloquium'
Trapizon/Istanbul 23.7-10.8/1979

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The main purpose of these lectures is to treat instantons and monopoles, as far as possible, as the same type of solutions of the Yang-Mills-Higgs field equations. For this reason, the only type of monopole solutions we consider are those for systems of fields where the self interaction potential of the Higgs fields vanishes, thus allowing the existence of self dual monopole solutions, in concert with the self dual instanton solutions, there being no known non self dual instantons.

Within our context, instantons and monopoles are special types of solutions to the classical equations of motion for gauge fields defined on four and three dimensional Euclidean base manifolds respectively. The special features of these solutions are that they have finite action, and that the value of this action isequal to a topological invariant via the condition of self duality.

The manner in which the qualitative unification of intantons and monopoles is tackled here is by way of developing a framework for the description of such solutions on N dimensional base manifolds. As Euclidean field equations these would be analogous to the static solutions of the Maxwell equations, namely the Laplace equation on N dimensional manifolds. Here we shall find that it is possible to extend the definition of instantons to any even dimensional manifold, and monopoles to any odd N.

We shall first give a brief introduction to classical gauge field theories by way of introducing our notation. Then the case of instantons and of monopoles will be treated separately. The last part will be devoted to the consideration of some explicit solutions as examples of the foregoing framework.

We stress again that we are only concerned with self dual solutions here, thereby not covering the well known non self dual monopole solutions of 't Hooft(1) and Polyakov(2).

GAUGE FIELDS

There are two types of fields in such theories. The gauge covariant fields, like the Higgs field, Dirac fiels etc., and the curvature or the second rank antisymmetric tensor fields, and the non gauge covariant fields called connections or vector potentials.

The elements of the gauge group are functions of the base manifold via the space dependence of the group parameter. The groups under consideration will be SU(n). The reason we do not take G the gauge group to be SU(2) for simplicity is, that on N dimensional manifolds with N>4 it will turn out that our new definitions of self duality will only allow non trivial solutions for G=SU(n) for n> 2+k with k> 1. In what follows the index of the base manifold will be labelled by a Greek index for even N, and a Latin index for odd N.

Also, because of our insistence on self dual solutions of the field equations, we shall consider only gauge covariant Higgs fields Φ that take their values in the algebra $\mathcal{G}$ of G. The Yang-Mills fields and $F_{\mu\nu}$ and potentials A_μ can take their values only in $\mathcal{G}$.

Under the action of an element g of G

$$g(x) = \exp \langle T,\theta(x)\rangle$$

where T are the (antihermitian) generators in and $\theta(x)$ the local parameters, gauge covariant fields transform as

$$\Phi \xrightarrow[g]{} g\Phi g^{-1} \tag{1a}$$

$$F_{\mu\nu} \xrightarrow[g]{} gF_{\mu\nu}g^{-1} \tag{1b}$$

while the Yang-Mills potential or connection is defined by the following transformation rule

$$A_\mu \xrightarrow[g]{} g A_\mu g^{-1} + g\partial_\mu g^{-1}. \tag{1c}$$

The transformation rule (1c) confirms our statement that A_μ must take its values in $\mathcal{G}$ since the inhomogeneous term $g\partial_\mu g^{-1}$ does take its values in $\mathcal{G}$ only.

It is clear from (1a) that $\partial_\mu\Phi$ is not gauge covariant, but that the covariant derivative

$$D_\mu\Phi = \partial_\mu\Phi + [A_\mu,\Phi]. \tag{2}$$

is covariant, by virtue of (1a,b,c), with

$$(D_\mu\Phi) \xrightarrow[g]{} g(D_\mu\Phi)g^{-1}. \tag{1d}$$

The Yang-Mills field, or the curvature tensor referred to above is in fact defined by the following relation

$$[D_\mu,D_\nu]\ f = F_{\mu\nu}f \tag{3a}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu,A_\nu] \tag{3b}$$

where f is a function taking its values in some representation of the group G.

An important consequence of the non commuting nature of the covariant derivative is the series of differential identities satisfied by the the curvature field, called the Bianchi identities.

In three and four dimensions, the Jacobi identity for the cyclic product of D_μ acting on some fuction f implies the following identities

$$\varepsilon_{ijk}\ [D_i,[D_j,D_k]] = 0 \tag{4a}$$

$$\varepsilon_{\mu\nu\rho\sigma}[D_\nu,[D_\rho,D_\sigma]] = 0 \tag{4b}$$

which lead, because of (3a), to the Bianchi identities

$$D_i{}^*F_i = D_\nu{}^*F_{\mu\nu} = 0. \tag{5a,b}$$

In (5) we have used the definition of the dual of the curvature

$${}^*F_i = \tfrac{1}{2}\ \varepsilon_{ijk}F_{jk} \tag{6a}$$

$${}^*F_{\mu\nu} = \tfrac{1}{2}\ \varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}\ . \tag{6b}$$

It is clear that in higher dimensional manifolds the curvature will satisfy further Bianchi identities in addition to(5), arising from the extended analogues of (4), fo example

$$\epsilon_{\mu_N\mu_{N-1}\cdots\mu_5\mu_4\mu_3\mu_2\mu_1}[D_{\mu_5},\{[D_{\mu_4},D_{\mu_3}],[D_{\mu_2},D_{\mu_1}]\}]=0 \quad (4b')$$

$$\epsilon_{\mu_N\mu_{N-1}\cdots\mu_7\cdots\mu_1}[D_{\mu_7},\{[D_{\mu_6},D_{\mu_5}],\{[D_{\mu_4},D_{\mu_3}],[D_{\mu_2},D_{\mu_1}]\}\}] = 0 \quad (4b'')$$

and so forth, giving rise to

$$\epsilon_{\mu_N\mu_{N-1}\cdots\mu_5\mu_4\mu_3\mu_2\mu_1}D_{\mu_5}\{F_{\mu_4\mu_3},F_{\mu_2\mu_1}\} = 0 \quad (5b')$$

$$\epsilon_{\mu_N\mu_{N-1}\cdots\mu_7\cdots\mu_1}D_{\mu_7}\{F_{\mu_6\mu_5},\{F_{\mu_4\mu_3},F_{\mu_2\mu_1}\}\} = 0 \quad (5b'')$$

and so forth.

It is useful at this stage to extend the duality operation (6b) to arbitrary dimensions. To this end we define the following curvature 2k forms

$$F(4)=F_{\mu\nu\rho\sigma}=\{F_{\mu\nu},F_{\rho\sigma}\} - \{F_{\mu\rho},F_{\nu\sigma}\} - \{F_{\mu\sigma},F_{\rho\nu}\} \quad (7)$$

$$F(6)=F_{\mu\nu\rho\sigma\tau\lambda}=\{F_{\mu\nu},F_{\rho\sigma\tau\lambda}\} - \{F_{\mu\rho},F_{\nu\sigma\tau\lambda}\} - \{F_{\mu\sigma},F_{\rho\nu\tau\lambda}\} - \{F_{\mu\tau},F_{\rho\sigma\nu\lambda}\} - \{F_{\mu\lambda},F_{\rho\sigma\tau\nu}\} \quad (7')$$

and so forth, noting that these 2k-forms have the same symmetries as the totally antisymmetric symbol.

Using these we extend the N=4 duality operation (6b) to the next two higher dimensions N=6 and 8

N=6
$$^{(+)}F_{\mu\nu} = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma\tau\lambda}F_{\rho\sigma\tau\lambda} \quad (8^+)$$

$$^{(-)}F_{\mu\nu\rho\sigma} = -\frac{i}{2!}\epsilon_{\mu\nu\rho\sigma\tau\lambda}F_{\tau\lambda} \quad (8^-)$$

N=8
$$^{(+)}F_{\mu\nu} = \frac{1}{6!}\epsilon_{\mu\nu\rho\sigma\tau\lambda\kappa\eta}F_{\rho\sigma\tau\lambda\kappa\eta} \quad (9^+)$$

$$^{(o)}F_{\mu\nu\rho\sigma} = \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma\tau\lambda\kappa\eta}F_{\tau\lambda\kappa\eta} \quad (9^o)$$

$$^{(-)}F_{\mu\nu\rho\sigma\tau\lambda} = \frac{1}{2!}\epsilon_{\mu\nu\rho\sigma\tau\lambda\kappa\eta}F_{\kappa\eta}\ . \quad (9^-)$$

It is important to note that for N=2p dimensional manifolds with odd p we have even number of duality operations and that $^{(\pm)}F$ are defined with factors of $\pm i$ respectively, while for manifolds with even p we have odd number of duality operations which do not involve any factors of i. We shall remark on the significance of of this property in the next Section.

These extended duality operations possess the following properties

$$^{(-)}[^{(+)}F(2)]=F(4),\quad ^{(+)}[^{(-)}F(4)]=F(2) \quad (10)$$

$$^{(-)}[^{(+)}F(2)]=F(6),\ ^{(o)}[^{(o)}F(4)]=F(4),\ ^{(+)}[^{(-)}F(6)]=F(2). \quad (11)$$

Finally, using the notation of (7),(8) and (9) the Bianchi identities for N=6 and 8 condense to the respective forms

$$D_\nu{}^{(+)}F_{\mu\nu} = 0,\quad D_\sigma{}^{(-)}F_{\mu\nu\rho\sigma} = 0 \quad (12)$$

$$D_\nu{}^{(+)}F_{\mu\nu} = 0,\ D_\sigma{}^{(o)}F_{\mu\nu\rho\sigma} = 0,\ D_\lambda{}^{(-)}F_{\mu\nu\rho\sigma\tau\lambda} = 0\ . \quad (12')$$

This completes the introduction of our notation for gauge fields on N dimensional manifolds. So far we have not considered the Euler-Lagrange equations or equivalently the action density of these fields. This will be the main item of interest in the next two Sections.

INSTANTONS:even N

Self dual solutions of the Yang-Mills equations for N=4 are the so called instantons[3]. In this case there is only one dual $*F={}^{(o)}F$ in our notation. The conventional action density

$$\mathcal{L}_4 = \mathrm{tr}\, F_{\mu\nu}F_{\mu\nu} \quad (13)$$

leads to the equations of motion

$$D_\nu F_{\mu\nu} = 0 \quad (14)$$

which would be identically satisfied if the solutions in question were to be required to be self dual

$$F_{\mu\nu} = {}^{(o)}F_{\mu\nu} \quad (15)$$

by virtue of the Bianchi identity (5b).

That such solutions should have finite action, proportional to a topological invariant with integer values (the Pontryagin number) can best be seen following the demonstration of BPST[3].

Consider the inequality

$$\mathrm{tr} \int (F_{\mu\nu} - {}^{(o)}F_{\mu\nu})^2 \, d_4x \geqslant 0, \tag{16}$$

which yields

$$\int \mathscr{L}_4 d_4x \geqslant \mathrm{tr} \int F_{\mu\nu} {}^{(o)}F_{\mu\nu} d_4x , \tag{16'}$$

the right side of which is equal to $4\pi^2$ times a normalisation factor times an integer, provided that the curvature vanishes at infinity in the following manner

$$A_\mu \xrightarrow[x\to\infty]{} g\,\partial_\mu g^{-1} \tag{17}$$

where $g(x)$ is some element of G.

It is then clear that the self duality condition (15) causes the inequality to become an equality, and then (16') expresses the fact that the total action corresponding to such a solution is equal to the finite quantity on the rigt.

The question now is whether the above argument can be modified to apply to Yang-Mills fields defined on N=2p dimensional manifolds (p>2). As in the previous Section, we shall demonstrate our point by explicit consideration of the cases p=3 and 4.

From the above argument of BPST for N=4, it is clear that the central agency which controls the behaviour of such solutions is the integer topological invariant, with (17) satisfied. So we start by writing down the topological invariants respectively

N=6,p=3: Chern Class[4]

$$q_6 = \frac{1}{DV_6} \epsilon_{\mu\nu\rho\sigma\tau\lambda} \mathrm{tr} \int F_{\mu\nu} F_{\rho\sigma} F_{\tau\lambda} \, d_6x \tag{18}$$

N=8,p=4: Pontryagin Class[4]

$$q_8 = \frac{1}{DV_8} \epsilon_{\mu\nu\rho\sigma\tau\lambda\kappa\eta} \int (\mathrm{tr}F_{\mu\nu}F_{\rho\sigma}F_{\tau\lambda}F_{\kappa\eta} - \tfrac{1}{2}\mathrm{tr}F_{\mu\nu}F_{\rho\sigma}.\mathrm{tr}F_{\tau\lambda}F_{\kappa\eta}) d_8x \tag{19}$$

where D is a normalisation factor depending on the representation matrices of $\mathbf{G}$ and V_N is the volume of the N dimensional sphere.

It is now clear that if we are to be able to use an argument similar to the BPST one, where the action integral becomes equal to the topological invariant, then we must expect the action densities to be of progressively higher order in the curvature as N increases.

Thus for N=6 let us consider the following inequality

$$\mathrm{tr} \int (F(2) - {}^{(+)}F(2))^2 d_6x \geqslant 0. \tag{20}$$

It is also possible to use the inequality

$$\mathrm{tr} \int (F(4) - {}^{(-)}F(4))^2 d_6x \geqslant 0$$

but this simply reduces to (20).

Expanding the integrand of (20), it is readily seen that the csoss term $\mathrm{tr}F(2)\,{}^{(+)}F(2)$ leads to the integral formula for the topological invariant given by (18). It is therefore obvious that if we were to consider the (unconventional) gauge theory with action density

$$\mathscr{L}_6 = \mathrm{tr}\,[F(2)^2 + {}^{(+)}F(2)^2] \quad , \tag{21}$$

and rewrite (20) as

$$\int \mathscr{L}_6 d_6x \geqslant 2DV_6 q_6 \quad , \tag{20'}$$

then, if we imposed the following extended self duality conditions

$$F(2) = {}^{(+)}F(2) \quad \text{or} \quad F(4) = {}^{(-)}F(4) \tag{22}$$

then (20') would become an equality, and the total action corresponding to these solutions will be finite.

Similarly for N=8, we consider the following inequality

$$\mathrm{tr} \int [F(2) - {}^{(+)}F(2)]^2 d_8x + \frac{1}{2} \int [\mathrm{tr}F(4) + {}^{(o)}F(4)]^2 d_8x \geqslant 0 \tag{23}$$

or the equivalent inequality obtained by replacing F(2) and ${}^{(+)}F(2)$ by F(6) and ${}^{(-)}F(6)$ respectively.

Then expanding the integrands in (23) we find

$$\int \mathcal{S}_8 d_8x \geqslant 2DV_8 q_8 \tag{24}$$

where the action density for the eight dimensional gauge theory is taken to be

$$\mathcal{S}_8 = \mathrm{tr}[F(2)^2 + {}^{(+)}F(2)^2] + 2[\mathrm{tr}F(4)]^2 . \tag{25}$$

The total action is then equal to the (finite) quantity $2DV_8 q_8$ when (24) becomes an equality subject to the extended self duality conditions

$$F(2) = {}^{(+)}F(2) \quad \text{or} \quad F(6) = {}^{(-)}F(6) \tag{26}$$

and

$$F(4) = -{}^{(o)}F(4). \tag{26'}$$

Two remarks are now in order. The first is that theself duality condition (22) shows clearly why we had anticipated a factor of i in the definition of the duality (8), for otherwise (22) would lead to a trivial solution $F(2)=0$. This situation will always occur for <u>odd</u> p ($N=2p$), in contrast with the cases of <u>even</u> p , where for exaple the duality (9) do not involve factors of i.

The second remark is that for $N>4$, the gauge group $G=SU(n)$ will allow non trivial solutions of (22) and (26)(26') only if $n>2$. It is easy to check for example that $F(2)=0$ for $n=2$ and $N=6$, by taking the trace of the self duality equation.

Finally it is clear that the extended self duality conditions solve the Euler-Lagrange equations arising from the corresponding action densities. For example, for $N=6$ the field equations are

$$D_\mu F_{\mu\nu} + \frac{1}{2} D_\mu \{F_{\mu\nu\rho\sigma}, F_{\rho\sigma}\} = 0 \tag{27}$$

which is identically satisfied if (22) hold by virtue of the Bianchi identities (12).

MONOPOLES:odd N

Our considerations here are restricted to self dual monopoles In three dimensions these are the solutions foûd by Prasad and Sommerfield[5] and by Bogomolnyi[6]. The Lagrangian density of these gauge theories include no self interaction term of the Higgs fields, the solutions however possess suitable boundary conditions which otherwise would have been consistent with and implied by the peculiar form of the Higgs potential[1].

We review the N=3 case briefly before going on to the N=5 case as an example of theextension of the notion of self duality for monopoles.

The Lagrange density for N=3

$$\mathcal{L}_3 = \frac{1}{4}\,\mathrm{tr}\, F_{ij}^2 + \frac{1}{2}\,\mathrm{tr}(D_i\phi)^2 \tag{28}$$

leads to the field equations

$$D_j F_{ij} + [\phi, D_i\phi] = 0 \tag{29}$$

$$D_i D_i \phi = 0 . \tag{30}$$

It is easy to verify that the 'self duality' condition

$$F_{ij} = \epsilon_{ijk} D_k \phi \tag{31}$$

solves (29) immediately, and reduces (30) to the Bianchi identity (5a). Therefore the problem of solving (29) and (30) is reduced to finding solutions of (31).

Now solutions of (31) that satisfy the following boundary conditions

$$D_i\phi \xrightarrow[r\to\infty]{} 0 \qquad \mathrm{tr}\,\phi^2 \xrightarrow[r\to\infty]{} \text{const.} \tag{32}$$

and

$$\phi \xrightarrow[r\to\infty]{} \text{const.}\,\mathbb{1} \tag{33}$$

have finite action (energy), equal to 4πtimes an integer.This is seen by considering the following inequality

$$\mathrm{tr} \int (F_{ij} - \epsilon_{ijk} D_k\phi)^2 d_3x \geqslant 0. \tag{34}$$

Expanding the integrand, and using Stokes' theorem and the Bianchi identity (5a), this reduces to

$$\int \mathcal{L}_3 d_3x \geqslant \epsilon_{ijk}\,\mathrm{tr}\int \phi F_{jk} dS_i , \tag{34'}$$

and this inequality becomes an equality if (31) is imposed.

It now remains to show that

$$\frac{1}{4\pi}\epsilon_{ijk}\,\mathrm{tr}\int \phi F_{jk} dS_i = \frac{1}{4\pi}\epsilon_{ijk}\,\mathrm{tr}\int \phi\partial_j\phi\partial_k\phi\, dS_i , \tag{35}$$

where the right side is a Kronecker integral subject to (32), and so takes on integer values only.

To arrive at (35) we introduce the so called 'electromagnetic' field

$$\mathcal{F}_{ij} = \mathrm{tr}\, \Phi(F_{ij} + \tfrac{1}{4}[D_i\Phi, D_j\Phi]) \qquad (36)$$

which on the $r \longrightarrow \infty$ surface over which (34') is integrated, is equal to the integrand of the right side of (34'), because of (32). But by using (33), we can show that (36) reduces to

$$\mathcal{F}_{ij} = \partial_i B_j - \partial_j B_i + \mathrm{tr}\, \Phi[\partial_i\Phi, \partial_j\Phi] \qquad (36')$$

with

$$B_i = \mathrm{tr}\, \Phi A_i \ .$$

This proves (35), since the surface integral of the pure curl term $\partial_i B_j - \partial_j B_i$ vanishes.

Therefore the energy integral is finite and equal to an integer modulo 4π, provided that (31),(32) and (33) hold.

We observe from the above that the integer bounding the action integral is a Kronecker integral. As there exist no global topological invariants on odd dimensional manifolds, we expect that the bounding integer should also be a Kronecker integral for higher dimensional Yang-Mills-Higgs theories.

We proceed to define the extended self duality conditions, that generalise (31), by considering the N=5 case as an example. The Kronecker integral for N=5 is given by

$$\frac{1}{4\pi^2}\epsilon_{ijklm}\, \mathrm{tr} \int \Phi\partial_i\Phi\partial_j\Phi\partial_k\Phi\partial_l\Phi\, dS_m \qquad (37)$$

in terms of the Higgs field satisfying (32).

As for the N=3 case, we define an 'electromagnetic' field

$$\mathcal{F}_{ijkl} = \mathrm{tr}\, \Phi\{(F_{ij} + \tfrac{1}{4}[D_i\Phi, D_j\Phi]), (F_{kl} + \tfrac{1}{4}[D_k\Phi, D_l\Phi])\} \quad (38)$$

which at large distances behaves, because of (32), like

$$\mathcal{F}_{ijkl} \xrightarrow[r \to \infty]{} \mathrm{tr}\, \Phi F_{ij} F_{kl} \ . \qquad (38')$$

On the other hand, using (33) it can be seen that

$$\mathcal{F}_{ijkl} = \tfrac{1}{2}\, \mathrm{tr}\, \Phi\partial_i\Phi\partial_j\Phi\partial_k\Phi\partial_l\Phi + \text{total divergence.} \qquad (38'')$$

We are now in a position to find a Lagrangian density in five dimensions, whose action integral is controlled by the integer (modulo $4\pi^2$) given by (37). To this end we consider the following inequality

$$\mathrm{tr} \int (\tfrac{1}{4!}\epsilon_{ijklm} F_{jklm} - D_i\Phi)^2 d_5x > 0. \qquad (39)$$

Choosing the Lagrangian density

$$\mathcal{L}_5 = \mathrm{tr}\, (\tfrac{1}{8} F_{ijkl}{}^2 + (D_i\Phi)^2) \qquad (40)$$

and expanding (39) we get

$$\int \mathcal{L}_5 d_5x \geq \frac{1}{4!}\epsilon_{ijklm}\, \mathrm{tr} \int \Phi F_{jklm} dS_i \qquad (39')$$

where in the last step we have used the Stokes' theorem and the Bianchi identity

$$\epsilon_{ijklm} D_i F_{jklm} = 0.$$

The right side of (39') is now obviously equal to $4\pi^2$ times (37), by virtue of (38),(38') and (38''), and the action integral of this five dimensional Yang-Mills-Higgs theory will be finite and proportional to an integer provided that the following self duality holds

$$\frac{1}{4!}\epsilon_{ijklm} F_{jklm} = D_i\Phi, \qquad (41)$$

which turns (39') into an equality. This is a direct generalisation of (31), for N=5.

Here we remark that a gauge theory with G=SU(n),N>3 has non trivial self dual solutions only if n>2. For example with N=5 and n=2 $D_i\Phi$ vanishes everywhere, as can be checked easily by taking the trace of (41).

That such solutions for N=3 were called monopole is because the integral in (35) can be written as

$$\int \tfrac{1}{2}\epsilon_{ijk} \mathcal{F}_{jk} dS_i = \int \vec{H}.d\vec{S},$$

which is the magnetic flux of a monopole.

EXAMPLES

We have so far not considered any explicit solutions of the above mentioned types. Such solutions have been found, starting with a suitble Ansatz for the form of the solutions and substituting these into the equations of motion, or equivalently the self duality equations. In this way the number of unknown functions is suitably reduced and the ensuing differential equations are solved. In fact all known exact solutions are of the self dual type. This is not surprising, for the self duality equations being of lower order than the equations of motion are much easier to solve. In addition for N=4 the self duality equations are also linearisable, while this is not the case for N=3 and probably not for higher N.

We shall not review any of the explicit instanton solutions here for these are very well known. They are the solutions of BPST[3], Witten[7] and Jackiw Nohl and Rebbi[8], all for G=SU(2). Exact instanton solutions for G=SU(3) are found by Bais and Weldon[9] and multi-instanton solutions for G=SO(5) by Tchrakian and Rawnsley[10].

Known exact monopole solutions on the other hand are much fewer, and only of the spherically symmetric variety. These are the solutions found by Prasad and Sommerfield[5] for G=SU(2) and by Bais and Weldon[11] for G=SU(N+1). Both these solutions are self dual in the sense described in the previous Section.

It is the Prasad-Sommerfield (P-S) solution that we wish to discuss in some detail here, because in reference(5) it has been arrived at from the second order equations of motion by a trial and error method, while here we shall derive it from the first order equations of duality systematically. We hope that our procedure may be of some help also in finding non spherically symmetric solutions.

We start by making the following not very specific Ansatz for the solutions of (29) and (30), or (31)

$$A_i = \epsilon_{ijk}\, \tau_j\, \alpha_m \tag{42a}$$

$$\Phi = \tau_i\, \beta_i \qquad ; \ \tau_i = -\tfrac{i}{2}\sigma_i . \tag{42b}$$

The spherically symmetric Ansatz of P-S is given in particular by

$$\vec{\alpha} = \frac{(K(r)-1)}{r^2}\vec{x} \ , \ \vec{\beta} = \frac{H(r)}{r^2}\vec{x} \tag{42'a,b}$$

whence the Euler-Lagrange equations (29),(30) yield

$$r^2K'' = K(K^2-1) + KH^2 \tag{29'}$$

$$r^2H'' = 2HK^2 \ . \tag{30'}$$

The solution to this system of second order coupled non linear differential equations was found , by a trial and error method, in reference(5) to be

$$K = \frac{kr}{\sinh kr} \ , \ H = kr \coth kr - 1 \ . \tag{43a,b}$$

In contrast the (anti)-self duality equations give the following first order coupled non linear differential equations

$$rK' = -HK \ , \ rH' = H - K^2 + 1 \tag{31a,b}$$

which are obviously satisfied by (43a,b). Although much simpler than (29'),(30'), these equations too do not lend themselves to a particularly obvious method of integration.

To proceed more systematically we substitute (42a,b) directly into the (anti)-self duality equation and have

$$\delta_{ij}\vec{\nabla}.\vec{\alpha} - \partial_j\alpha_i + \alpha_i\alpha_j + \partial_i\beta_j + \vec{\alpha}.\vec{\beta}\,\delta_{ij} - \beta_i\alpha_j = 0 \ . \tag{31'}$$

This system of equations contains six unknown functions $(\vec{\alpha},\vec{\beta})$ instead of $(\vec{A},\vec{\Phi})$ and is therefore not more specific than (31). A more specific Ansatz would be

$$\vec{\alpha} = \vec{\nabla} \ln \rho \ , \ \vec{\beta} = \vec{\nabla} \ln \Omega\rho \ , \tag{42''a,b}$$

leading to

$$\delta_{ij}(\rho^{-1}\Delta\rho + \vec{\nabla}\ln\rho.\vec{\nabla}\ln\Omega) + \partial_i\partial_j\ln\Omega - \partial_i\ln\rho\ \partial_j\ln\Omega = 0, \tag{31''}$$

involving now only two unknown functions.

At this stage it looks as if (31'') might lead to multi-monopole solutions, if we were to make the further Ansatz that $\Omega(x,y,z) = 1$, whence (31) would reduce to

$$\rho^{-1}\Delta\rho = 0$$

that is

$$\rho = 1 + \sum_n \frac{\lambda_n^2}{|\vec{x}-\vec{y}_n|} \quad , \tag{44}$$

leading to the monopole charge (which through self duality is proportional to the energy) according to (35)

$$\mu = \frac{1}{4\pi} \int d_3x \ \Delta\Delta \ln\rho \quad . \tag{45}$$

Unfortunately the integrand in (45) is singular at the origin and blows up, so (44) is <u>not</u> a 'monopole' solution. This is in contrast with the multi-instanton[8] where the singularities of the field, due to the singularities of the solutions of the four dimensional Laplace equation, can be gauged away.

Here the corresponding singularities in A_i arising from those in (44) cannot be gauged away. For example taking the first term in the sum in (44), ithave

$$\vec{A} = \frac{1}{2} \frac{|z|}{|z|+\lambda^2} g^{-1}\vec{\nabla}g - \frac{1}{2} g^{-1}\vec{\nabla}g \tag{46}$$

with

$$\vec{z} = \vec{x}-\vec{y}_1 \ , \quad g = -2\frac{\vec{z}.\vec{\tau}}{|z|} \in SU(2)$$

and under a gauge transformation with the element g of G

$$A_i = \varepsilon_{ijk} \tau_j \frac{\hat{z}_k}{|z|+\lambda^2} + \frac{3}{2} g\partial_i g^{-1} \tag{46'}$$

where the singular (second) term has not been removed, unlike the multi-instanton case.

We now go back to the Ansatz (42'') and as a simplification require that both ρ and Ω depend only on the radial variable r.

Introducing then the function

$$\xi(r) = \frac{d}{dr} \ln\Omega$$

we find from (31'') that

$$\rho = r^{-1}\xi \tag{47}$$

and that $\xi(r)$ is given by

$$\frac{d\xi}{dr} + \frac{1}{2}\xi^2 = \text{const.} \tag{48}$$

which can be integrated immediately. Choosing the constant on the right side of (48) to be positive or negative, say $\pm\frac{1}{2}k^2$,

we have respectively the following two solutions

$$\xi = k \tanh \tfrac{1}{2}kr \tag{49}$$

$$\xi = -k \tan \tfrac{1}{2}kr. \tag{49'}$$

The solution (49) is precisely the P-S solution, which has here been derived systematically.

Solution (49') on the other hand does not satisfy the criteria for it to be a 'monopole'. In this case, even though A, and ϕ are themselves regular at the origin, the Lagrange ~~density~~ integral constructed from them is not converge. This ~~singular~~ divergent behaviour is reflected in the fact that (49') does not respect the finite energy condition (32), and the magnitude of the Higgs field oscillates at large distances. The corresponding mmagnetic charge and energy are infinite.

There is another non'monopole' solution of (48),

$$\xi = \frac{2}{r+a} \quad . \tag{50}$$

Both the Yang-Mills potential and the Higgs field are singular at the origin, the finite energy condition (32) is not satisfied, and the energy integral diverges but just logarithmically.

The only instructive aspect of this solution is that the resulting connection field satisfies the 'instanton' boundary condition (17)

$$\vec{A} \xrightarrow[r\to\infty]{} g^{-1}\vec{\nabla}g \quad ,$$

with

$$g = -\frac{2\vec{x}.\vec{\tau}}{r} \quad ,$$

pointing out to the fact that on a three dimensional manifold there are no 'instanton' type solutions of the Yang-Mills-Higgs equations.

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