DIAS-TP-78-01

A non-unitary B-K-S pairing of polarizations
(Revised version)

J. H. Rawnsley

School of Theoretical Physics
Dublin Institute for Advanced Studies
Dublin 4, Ireland

Abstract

The half-form pairing of two polarizations of the Kepler manifold is found and shown to define a bounded linear isomorphism of the two Hilbert spaces, but is not unitary.

1. Introduction

In [15] J.-M. Souriau showed that, when suitably completed, the phase space and flow of the Kepler problem in n-dimensions could be identified with $T_0^* S^n$ (the cotangent bundle of the n-sphere minus its zero section), and its geodesic flow (for the standard metric). This extended a similar result of J. Moser [7] concerning the energy surfaces. Souriau also observed that $T_0^* S^n$ had a complex structure invariant under the flow. In [10] I showed this complex structure was a positive polarization for the natural symplectic structure of the cotangent bundle and therefore determines a quantization of the flow [6, 13, 14].

$T_0^* S^n$ has a real polarization, given by the cotangent fibres, but this is not invariant under the flow. By using the method of moving polarizations, J. Elhadad [3] quantized the flow using a limiting procedure, despite an obstruction to the formal pairing noticed by R. Blattner [2]. There is no obstruction to the pairing of the real and complex polarizations, so we can use the transformation defined by the pairing [2, 5, 6] to carry the quantization of the flow from the complex to the real polarization. The generator of the unitary group so obtained on $L^2(S^n)$ is $2\pi[-\Delta + (n-1)^2/4]^{1/2}$ which has spectrum $2\pi(k + (n-1)/2)$, $k = 0, 1, 2, \ldots$ This agrees with the semi-classical spectrum of A. Weinstein [16] but has different multiplicities.

The pairing of these two polarizations is of interest since it is not unitary. It requires some tedious computations to establish it as a bounded linear operator between the Hilbert spaces of the two polarizations. It is closely related to the Laplace representation of spherical harmonics [8].

This paper is divided up as follows: §2 summarizes the theory of polarizations and half-form pairings and as an example I obtain Bargmann's transform [1] between the real and complex polarizations of $\mathbb{R}^{2n} \cong \mathbb{C}^n$. The real and complex polarizations of $T_0^* S^n$ together with the formal expression for their pairing are described in §3. The rigorous existence and non-unitary nature of the pairing is established in §4. An appendix contains the evaluation of some integrals required in §4.

I thank the referee for suggestions which have considerably improved the presentation of the original manuscript. Thanks are also due to R. Blattner and D. Simms for their help and interest in this work.

2. Polarizations and the half-form pairing.

If (X, ω) is a symplectic manifold, the space $C(X)$ of complex functions on X is a Lie algebra under Poisson bracket:

$$[\varphi, \psi] = \xi_\varphi \psi \quad ; \quad \xi_\varphi \lrcorner\, \omega = d\varphi .$$

If ω determines an integral de Rham cohomology class, there is a Hermitian line bundle L with connection ∇ over X having curvature $2\pi i \omega$. The space ΓL of sections of L is a $C(X)$-module where, for $\varphi \in C(X)$, $s \in \Gamma L$

$$\varphi \cdot s = \nabla_{\xi_\varphi} s + 2\pi i \varphi s .$$

This representation of $C(X)$ is known as prequantization. See [4] for details.

A polarization of (X, ω) is a subbundle F of the complexified tangent bundle $TX^{\mathbb{C}}$ which is

(i) isotropic;

(ii) maximal with respect to (i);

(iii) integrable.

Condition (i) means ω vanishes identically when restricted to F. If $\dim X = 2n$, then by (ii) $\dim F_x = n$ for all $x \in X$. If $F^\circ \subset T^*X^{\mathbb{C}}$ denotes the bundle of covectors vanishing on F, then (i) and (ii) are equivalent to $\xi \longmapsto \xi \lrcorner\, \omega$ maps F isomorphically onto F°. We shall take integrable to mean: $F \cap \overline{F}$ has constant dimension and F, $F + \overline{F}$ are closed under the Lie bracket of vector fields. Thus the complex Frobenius theorem of Nirenberg [8] applies to F.

There are two main examples of polarizations. *If* $F = \overline{F}$, F *is called* real and is the tangent bundle of a Lagrangian foliation of (X, ω). The fibres of a cotangent bundle $X = T^*M$ is a typical example of this situation. At the other extreme we may have $F \cap \overline{F} = 0$, in which case $TX^{\mathbb{C}} = F \oplus \overline{F}$ so that an almost complex structure J may be defined on X in such a way that F

consists of tangents of type $(0,1)$. Since F is involutive, J is integrable and X becomes a complex manifold.

$$g(\xi,\eta) = \omega(J\xi,\eta)\ ; \quad \xi,\eta \in \Gamma TX$$

defines a non-singular symmetric bilinear form on the tangent spaces to X which is Hermitian for the complex structure. The associated 2-form is ω which is closed, so that g is a (pseudo-) Kaehler metric. Thus any Kaehler manifold is an example of a symplectic manifold with a polarization.

If F is a polarization of (X,ω) it is called positive if

$$-i\,\omega(\xi,\bar{\xi}) \geq 0\,, \quad \forall\ \xi \in \Gamma F.$$

Real polarizations are always positive, whilst if $F \cap \bar{F} = 0$, F is positive if and only if g is positive definite.

Given a polarization F of (X,ω) we can define the structure sheaf $\mathcal{C}_F$ as the sheaf associated to the presheaf

$$U \mapsto C_F(U) = \{\varphi \in C(U) \mid \xi\varphi = 0,\ \forall \xi \in \Gamma F\}, \quad U \subset X \ \text{open.}$$

See [6, 12] for some properties of this sheaf. When $F \cap \bar{F} = 0$, $\mathcal{C}_F$ is the sheaf of holomorphic functions on X.

Let L,∇ be a prequantization of (X,ω) and F a polarization, then we set

$$\Gamma_F L = \{s \in \Gamma L \mid \nabla_\xi s = 0,\ \forall \xi \in \Gamma F\}.$$

$\Gamma_F L$ is not stable under all $\varphi \in C(X)$, but those functions φ which preserve $\Gamma_F L$ form a Lie subalgebra $C^1_F(X)$ which contains $C_F(X)$ as a maximal abelian ideal. The representation of $C^1_F(X)$ on $\Gamma_F L$ is called the quantization with respect to F.

If $U \subset X$ is open with $H^1(U,\mathcal{C}_F) = 0$ and $\omega|U = d\Theta$ with $\Theta|F = 0$ then there is a nowhere vanishing section s of L over U with $\nabla_\xi s = 2\pi i \Theta(\xi) s$ for all vector fields ξ. $\Gamma_F(L|U)$ can be identified with $C_F(U)$ by $\varphi \mapsto \varphi s$, $\varphi \in C_F(U)$ and if $\psi \in C^1_F(U)$

$$\psi.(\varphi s) = \{[\psi,\varphi] + 2\pi i(\Theta(\xi_\psi) + \psi)\varphi\}s.$$

In general it is difficult to make $\Gamma_F L$ into a Hilbert space, which is desirable if this construction is going to be used to construct the quantum mechanical model corresponding with the classical system described by (X,ω). Even when this is possible there is no way of comparing $\Gamma_F L$ with $\Gamma_G L$ for different polarizations F and G. For these reasons B. Kostant introduced the notion of half-forms and their pairing in [5, 6], and this was further developed by R. Blattner [2]. There is no satisfactory theory at present unless F and G are both positive. The formalism we shall use is that of [11].

If F is a polarization of (X,ω), dim $X = 2n$, then $\Lambda^n F^0$ is a line bundle, the canonical bundle K^F of F. If $F \cap \bar{F} = 0$, K^F is the canonical bundle of the complex structure. For F positive the Chern class of K^F is determined by ω so that K^F and K^G are isomorphic as C^∞ line bundles for any two positive polarizations F and G. In this case $K^F \otimes \overline{K^G}$ is trivial, and a pairing of K^F with K^G is a choice of a trivialization of this bundle.

When $F \cap \bar{G} = 0$ exterior multiplication defines an isomorphism of $K^F \otimes \overline{K^G}$ with $\Lambda^{2n} T^*X^{\mathbb{C}}$ and the latter is trivialized by the Liouville volume $\lambda = (-1)^{n(n-1)/2}\omega^n/n!$. Hence if $\alpha \in \Gamma K^F$, $\beta \in \Gamma K^G$ we define $\langle\alpha,\beta\rangle$ by

$$i^n \langle\alpha,\beta\rangle\,\lambda = \alpha \wedge \bar{\beta}.$$

If $F \cap \bar{G}$ has constant rank then $F \cap \bar{G} = D^{\mathbb{C}}$ for a real integrable isotropic subbundle D of TX (positivity of F and G is required here).

Let $D^\perp$ denote all $\xi \in TX$ with $\omega(\xi, D) = 0$, then $D \subset D^\perp$ and ω induces a non-singular skew form ω/D on $D^\perp/D$ making $D^\perp/D$ a symplectic vector bundle. Since $D \subset F$, $F \subset (D^\perp)^{\mathbb{C}}$ so projects to give a maximal isotropic subbundle F/D of $(D^\perp/D)^{\mathbb{C}}$. The same is true of G, and $F/D \cap \overline{G/D} = 0$. Then $K^{F/D}$ and $K^{G/D}$ are paired by exterior multiplication as above. We lift this pairing to K^F and K^G as follows:

Let $b = (e_1, \cdots, e_k)$ be a frame for D_x. Then it can be extended to a frame $(e_1, \ldots, e_k, f_1, \ldots, f_{n-k})$ for F_x and if $\alpha \in K^F_x$,

$$\alpha = a\,(e_1 \lrcorner\, \omega)_\wedge \cdots {}_\wedge (e_k \lrcorner\, \omega)_\wedge (f_1 \lrcorner\, \omega)_\wedge \cdots {}_\wedge (f_{n-k} \lrcorner\, \omega)$$

for some $a \in \mathbb{C}$. Let $\tilde{f}_i$ be the projection of $f_i \in (D^\perp_x)^{\mathbb{C}}$ into $(D^\perp/D)^{\mathbb{C}}_x$ so that $(\tilde{f}_1, \cdots, \tilde{f}_{n-k})$ is a frame for $(F/D)_x$. Put

$$\tilde{\alpha}_b = a\,(\tilde{f}_1 \lrcorner\, \omega/D)_\wedge \cdots {}_\wedge (\tilde{f}_{n-k} \lrcorner\, \omega/D) \in K^{F/D}_x.$$

Then $\tilde{\alpha}_b$ does not depend on the extension $f_1, \ldots, f_{n-k}$ and if $g \in GL(k, \mathbb{R})$,

$$\tilde{\alpha}_{b \cdot g} = \mathrm{Det}[g^{-1}]\, \tilde{\alpha}_b.$$

We can project $\beta \in K^G_x$ in the same fashion. Put

$$\langle \alpha, \beta \rangle (b) = \langle \tilde{\alpha}_b, \tilde{\beta}_b \rangle.$$

Then $\langle \alpha, \beta \rangle$ is a density of order -2 on D and using the Liouville density on TX defines a density of order 2 on $(TX)/D$.

Let us suppose the space X/D of leaves of the foliation D is smooth then $(TX)/D$ is the pull back to X of the tangent bundle $T(X/D)$. If $\langle \alpha, \beta \rangle$ is covariant constant along the leaves it will project down to a density of order 2 on X/D. If we could everywhere take a square root we should end with a density of order 1 on X/D which would be a candidate for integrating over X/D to obtain a global pairing.

There are clearly many points at which this procedure can break down. First, K^F may not have a square root. It has one precisely when its Chern class is divisible by 2 (in which case (X, ω) is called metaplectic). Assuming this is so, the symplectic frame bundle of (X, ω) has a double covering from which a square root Q^F of K^F can be canonically constructed for each positive polarization F. These square roots have the property that $Q^F \otimes \overline{Q^G}$ is trivial, which is necessary if a pairing is to exist. See [2] for the construction. Sections of Q^F are called half-forms normal to F.

There is a pairing $\langle \cdot, \cdot \rangle$ of $Q^F \otimes \overline{Q^G}$ into the densities of order -1 on D such that for $\mu \in \Gamma Q^F$, $\nu \in \Gamma Q^G$,

$$\langle \mu, \nu \rangle^2 = \langle \mu \otimes \mu, \nu \otimes \nu \rangle.$$

The procedure now is to replace L by $L \otimes Q^F$, and define $\Gamma_F L \otimes Q^F$ by introducing a covariant derivative in Q^F. It is fortunate that Q^F has a covariant derivative along F arising from Lie differentiation in K^F. If $\xi \in \Gamma F$, $\alpha \in \Gamma K^F$ then

$$\nabla_\xi \alpha = \xi \lrcorner\, d\alpha$$

defines ∇_ξ in ΓK^F and

$$\nabla_\xi (\mu_1 \otimes \mu_2) = (\nabla_\xi^{1/2} \mu_1) \otimes \mu_2 + (\mu_1 \otimes \nabla_\xi^{1/2} \mu_2)$$

defines $\nabla_\xi^{1/2}$ uniquely in ΓQ^F. Then $\nabla \otimes 1 + 1 \otimes \nabla^{1/2}$ defines a connection along F in $L \otimes Q^F$ and $\Gamma_F L \otimes Q^F$ is defined as before.

$\Gamma_F L \otimes Q^F$ is paired with $\Gamma_G L \otimes Q^G$ by pairing L with itself using the Hermitian structure and Q^F with Q^G using $\langle\, ,\rangle$. Lie differentiation defines a connection along D in the densities on $(TX)/D$, but Blattner found that, in general. $\nabla_\xi\langle\rho,\sigma\rangle$ need not vanish for $\rho\in\Gamma_F L\otimes Q^F$, $\sigma\in\Gamma_G L\otimes Q^G$. In all the cases we are interested in $\langle\rho,\sigma\rangle$ does project to a density on X/D so we shall not investigate this point further.

To obtain the inner product in $\Gamma_F L\otimes Q^F$ one pairs F to itself. Let $\mathcal{H}_F$ be the resulting Hilbert space (which may consist only of zero). If $F\cap\overline{F}=D^{\mathbb{C}}$ the inner product involves integrating over X/D. If $F\cap\overline{F}=0$ this is integration over X.

As an example take $X=\mathbb{R}^{2n}$, $\omega=\sum_{i=1}^{n}dp_i\wedge dq_i$ where we take $(q_1,\dots,q_n,p_1,\dots,p_n)$ as coordinates. Then F, spanned by $\partial/\partial p_1,\dots,\partial/\partial p_n$, is a real polarization and K^F is spanned by $dq_1\wedge\cdots\wedge dq_n$ so is trivial. Let Q^F be spanned by $(dq_1\wedge\cdots\wedge dq_n)^{1/2}$ (defined up to a global sign). If $\theta=\sum_{i=1}^{n}p_i\,dq_i$, θ vanishes on F and $\omega=d\theta$. If L,∇ is a pre-quantization of (X,ω), L has a nowhere vanishing section s_0 with $\nabla_\xi s_0 = 2\pi i\,\theta(\xi)s_0$. Also $dq_1\wedge\cdots\wedge dq_n$ is closed, so $\nabla_\xi^{1/2}(dq_1\wedge\cdots\wedge dq_n)^{1/2}=0$ for all $\xi\in\Gamma F$. Thus $\Gamma_F L\otimes Q^F$ has elements of the form $\varphi s_0\otimes(dq_1\wedge\cdots\wedge dq_n)^{1/2}$ with $\partial\varphi/\partial p_i=0$, $i=1,\dots,n$. Thus φ is a function of $q_1,\dots,q_n$ only. Then s_0 can be normalized so that $|s_0|^2=1$, and $\langle(dq_1\wedge\cdots\wedge dq_n)^{1/2},(dq_1\wedge\cdots\wedge dq_n)^{1/2}\rangle$ projects to the density $dq_1\cdots dq_n$ on $\mathbb{R}^n$. Thus

$$\|\varphi s_0\otimes(dq_1\wedge\cdots\wedge dq_n)^{1/2}\|^2=\int_{\mathbb{R}^n}|\varphi(q_1,\dots,q_n)|^2\,dq_1\cdots dq_n.$$

In this case then, $\mathcal{H}_F=L^2(\mathbb{R}^n)$.

A second polarization G arises from the identification $\mathbb{R}^{2n}=\mathbb{C}^n$. Put $z_j=q_j+ip_j$, $j=1,\dots,n$ and let G be spanned by $\partial/\partial\bar z_1,\dots,\partial/\partial\bar z_n$. Then K^G is spanned by $dz_1\wedge\cdots\wedge dz_n$ and Q^G by $(dz_1\wedge\cdots\wedge dz_n)^{1/2}$. Let $\theta'=i/2\sum_{j=1}^{n}\bar z_j\,dz_j$ so that $\omega=d\theta'$ and θ' vanishes on G. We have a nowhere vanishing section t_0 of L with $\nabla_\xi t_0 = 2\pi i\,\theta'(\xi)t_0$. Then $t_0=\varphi_0 s_0$ for some nowhere vanishing function φ_0. According to [4], φ_0 is given by

$$d\log\varphi_0 = 2\pi i(\theta'-\theta)$$

which may be solved to give

$$\varphi_0 = \exp\{-\pi|z|^2/2-i\pi\sum_{j=1}^{n}q_jp_j\}.$$

Then $|t_0|^2=|\varphi_0|^2=\exp-\pi|z|^2$. Any element $t\in\Gamma_G L\otimes Q^G$ has the form $t=\psi t_0\otimes(dz_1\wedge\cdots\wedge dz_n)^{1/2}$ with ψ holomorphic, and since $(dz_1\wedge\cdots\wedge dz_n)\wedge\overline{(dz_1\wedge\cdots\wedge dz_n)}=(2i)^n\lambda$, we obtain $\langle(dz_1\wedge\cdots\wedge dz_n)^{1/2},(dz_1\wedge\cdots\wedge dz_n)^{1/2}\rangle=|\lambda|$ and so

$$\|t\|^2 = \int_{\mathbb{R}^{2n}}|\psi(z_1,\dots,z_n)|^2\exp-\pi|z|^2\,|\lambda|.$$

It follows $\mathcal{H}_G$ may be identified with the holomorphic functions on $\mathbb{C}^n$ square integrable for the Gaussian measure $\exp-\pi|z|^2\,|\lambda|$.

These polarizations F and G on $\mathbb{R}^{2n}$ are easily paired since $F\cap\overline{G}=0$ and $(dq_1\wedge\cdots\wedge dq_n)\wedge\overline{(dz_1\wedge\cdots\wedge dz_n)}=(i)^n\lambda$ so that $\langle(dq_1\wedge\cdots\wedge dq_n)^{1/2},(dz_1\wedge\cdots\wedge dz_n)^{1/2}\rangle=1$. Hence

$$\langle\varphi s_0\otimes(dq_1\wedge\cdots\wedge dq_n)^{1/2},\ \psi t_0\otimes(dz_1\wedge\cdots\wedge dz_n)^{1/2}\rangle =$$

$$\int_{\mathbb{R}^{2n}}\varphi(q_1,\dots,q_n)\overline{\psi(z_1,\dots,z_n)}\exp\{-\pi|z|^2/2+i\pi p\cdot q\}\,|\lambda|.$$

As a map from $\mathcal{H}_G$ to $\mathcal{H}_F$ this is formally given by

$$(T\psi)(q)=\int_{\mathbb{R}^n}\psi(q+ip)\exp\{-\pi(p^2+q^2)/2-i\pi p\cdot q\}\,d^np.$$

If ψ is a polynomial, it is in $\mathcal{H}_G$ and $T\psi\in\mathcal{H}_F$. Since polynomials are dense in $\mathcal{H}_G$, T is densely defined. Proving T is unitary is messy using polynomials, so instead we use that $\mathcal{H}_G$ has a reproducing kernel.

If $\psi_w(z)=\exp\pi\bar w\cdot z$, $\psi_w\in\mathcal{H}_G$ for all $w\in\mathbb{C}^n$, $(\|\psi_w\|^2=\exp\pi|w|^2)$, and for any $\psi \in \mathfrak{H}_G$,

$$\psi(w) = (\psi, \psi_w).$$

Then finite linear combinations $\sum_\alpha c_\alpha \psi_{w_\alpha}$ are dense in $\mathfrak{H}_G$ also, so we need only compute $T\psi_w$. This is a Gaussian integral and can be computed explicitly:

$$(T\psi_w)(q) = 2^{n/2} \exp\{-\pi q^2 - \pi \bar{w}^2/2 + 2\pi \bar{w}\cdot q\}.$$

Again $(T\psi_w, T\psi_v)$ is a Gaussian integral and may be evaluated as

$$(T\psi_w, T\psi_v) = \exp \pi v.\bar{w} = (\psi_w, \psi_v).$$

Thus T is an isometry on the dense domain above. If it has dense range it extends to a unitary map of $\mathfrak{H}_G$ onto $\mathfrak{H}_F$. That the range is dense follows because $(T\psi_w)(q)$ is essentially the generating function for the Hermite functions whose linear combinations are dense in $L^2(\mathbb{R}^n)$.

Using the reproducing kernel,

$$(T\psi)(q) = \int_{\mathbb{R}^n} (\psi, \psi_{q+ip}) \exp\{-\pi(p^2+q^2)/2 - i\pi p.q\}\, d^n p$$

$$= \int_{\mathbb{R}^{2n}} \psi(z)\, K(z,q) \exp -\pi|z|^2 \; |\lambda|,$$

with

$$K(z,q) = (T\psi_z)(q).$$

Apart from normalization, K is Bargmann's transform [1] from $\mathfrak{H}_G$ to $\mathfrak{H}_F$.

§3. The real and complex polarizations of $T_0^*S^n$

$T_0^*S^n$ can be identified with $X = \{(e,x) \in \mathbb{R}^{n+1}\times\mathbb{R}^{n+1} \mid e.e=1, x.e=0, x\neq 0\}$. The natural symplectic structure on $T_0^*S^n$ carries over to ω on X where $\omega = d\theta$, $\theta = x.de$, regarding the components $e_0,\dots,e_n,x_0,\dots,x_n$ as functions on X. For $n \geqslant 3$, X is simply-connected but $\pi_1(X) = \mathbb{Z}_2$ if $n=2$. To avoid technical complications arising from non-simple-connectedness we shall assume $n \geqslant 3$.

X fibres over $S^n = \{e \in \mathbb{R}^{n+1} \mid e.e = 1\}$ and the fibres are the cotangent spaces with the origin deleted. Put $\pi(e,x) = e$.

Let $|x| = (x.x)^{1/2}$, $h(e,x) = 2\pi|x|$, then $h \in C(X)$ and ξ_h generates a flow σ_t which may be found to be

$$\sigma_t(e,x) = (\cos 2\pi t\, e + \sin 2\pi t\, x/|x|\, ,\ \cos 2\pi t\, x - \sin 2\pi t\, |x| e).$$

This may be more neatly expressed by introducing $z \in \mathbb{C}^{n+1}$ with

$$z = |x|\, e + i x \tag{1}$$

and then

$$\sigma_t z = \exp -2\pi i t\, z.$$

$(e,x) \mapsto z$ injects X into $\mathbb{C}^{n+1}$ and the image is the non-singular cone

$$\{z \in \mathbb{C}^{n+1} \mid z.z = 0,\ z \neq 0\},$$

giving X a complex structure. Let $d = \partial + \bar\partial$ be the usual decomposition of the exterior derivative into components of type (1,0) and (0,1).

Of course $\bar\partial z_i = 0$, $i = 0,\dots,n$. From (1) $z.\bar{z} = 2|x|^2$ so

$$4|x|\partial|x| = 2\partial|x|^2 = \bar{z}.dz = 2|x|\,d|x| - 2i|x|\, x.de.$$

Thus $\Theta = i\partial|x| - i\bar\partial|x|$ and hence

$$\omega = 2i\,\bar\partial\partial|x|. \qquad (2)$$

This shows that ω is the Kaehler 2-form of a positive definite Hermitian metric and hence that the tangents of type (0,1) form a positive polarization G with $G\cap\bar G = 0$. Let $F = \mathrm{Ker}\,\pi_*$ be the tangent spaces to the fibering $\pi: X\to S^n$. Since $\sigma_t{}_* G = G$, $h\in C^1_G$. However, $h\notin C^1_F$ (though $h^2\in C^2_F$).

Let L, ∇ be a prequantization of (X,ω). Then $\omega = d\Theta$ implies the existence of a nowhere vanishing section s_F with $\nabla_\xi s_F = 2\pi i\,\Theta(\xi)\,s_F$. Θ is real so $|s_F|^2$ is constant and s_F can be normalized so $|s_F|^2 = 1$.

Similarly $\omega = d(2i\partial|x|)$ so we have s_G with $\nabla_\xi s_G = -4\pi\partial|x|(\xi)\,s_G$. But $s_G = \varphi_0 s_F$ for some nowhere vanishing function φ_0 and

$$d\log\varphi_0 = 2\pi i(2i\partial|x| - \Theta) = -2\pi\, d|x|$$

so

$$\varphi_0 = \exp -2\pi|x|$$

apart from a constant which we can set equal to 1. Thus $|s_G|^2 = |\varphi_0|^2 = \exp -4\pi|x|$. This completes the analysis of the prequantization.

To discover whether half-forms exist, consider K^F. Let ρ be any n-form on S^n then $\pi^*\rho$ is an n-form vanishing on F so $\pi^*\rho\in K^F$. Since S^n is orientable we can choose ρ nowhere vanishing, and then $\pi^*\rho$ vanishes nowhere, showing K^F is trivial. Thus there is a square root Q^F, unique since X is simply-connected. The same conclusion could have been reached from [5] since it is known that when F is the tangent bundle to a projection $\pi: X\to Y$ the mod 2 reduction of the Chern class of F is the square of the first Stiefel-Whitney class of Y, pulled back to X. Then, if Y is orientable, the Chern class must be even.

Observe also that since ρ is a form of maximum degree on S^n, $d\rho = 0$ so that $d\pi^*\rho = 0$ and hence $\nabla_\xi\pi^*\rho = 0$, $\xi\in\Gamma F$. Fix ρ_0 as the Riemannian volume on S^n, which in terms of the functions e_i is

$$\rho_0 = \sum_{j=0}^{n} (-1)^j e_j\, de_0\wedge\dots\wedge\widehat{de_j}\wedge\dots\wedge de_n \qquad (3)$$

where $\widehat{de_j}$ means that term is omitted. On the set where $e_k\neq 0$ we can take $e_0,\dots,e_{k-1},e_k,\dots,e_n$ as coordinates and obtain

$$\rho_0 = (-1)^k e_k^{-1}\, de_0\wedge\dots\wedge\widehat{de_k}\wedge\dots\wedge de_n. \qquad (4)$$

Expression (3) makes sense on X and gives $\pi^*\rho_0$.

Let $Q^F\otimes Q^F = K^F$ and μ_F be a section of Q^F with $\mu_F\otimes\mu_F = \pi^*\rho_0$, which exists since X is simply-connected. Then also $\nabla^{1/2}_\xi\mu_F = 0$ for all ξ in ΓF.

K^G may be handled similarly. We look for a section σ which has an expression analogous to (3) in terms of the functions z_i instead of e_i, and in order that $d\sigma = 0$ one finds

$$\sigma = |x|^{-2}\sum_{j=0}^{n}(-1)^j \bar z_j\, dz_0\wedge\dots\wedge\widehat{dz_j}\wedge\dots\wedge dz_n.$$

If $U_j\subset X$ is the subset where $e_j\neq 0$, then $z_j\neq 0$ on U_j and

$$\sigma|U_j = 2(-1)^j z_j^{-1}\, dz_0\wedge\dots\wedge\widehat{dz_j}\wedge\dots\wedge dz_n \qquad (5)$$

Thus σ vanishes nowhere and $\nabla_\xi\sigma = 0$, $\xi\in\Gamma G$. Let $Q^G\otimes Q^G = K^G$ and μ_G be the section of Q^G with $\mu_G\otimes\mu_G = \sigma$, so that $\nabla^{1/2}_\xi\mu_G = 0$ for ξ in ΓG.

We have thus shown $\Gamma_F L\otimes Q^F$ consists of sections of the form $\varphi\circ\pi\, s_F\otimes\mu_F$ with $\varphi\in C^\infty(S^n)$, and $\Gamma_G L\otimes Q^G$ of the form $\psi\, s_G\otimes\mu_G$

with Ψ holomorphic. The norms are easily computed as in §2. F is real so $\langle \mu_F, \mu_F \rangle = \pi^* |\rho_0|$, and

$$\|\varphi\|_F^2 = \int_{S^n} |\varphi|^2 |\rho_0| ,$$

so $\mathcal{H}_F$, the completion of $\Gamma_F L \otimes Q^F$ coincides with $L^2(S^n, |\rho_0|)$.

For G we have $G \cap \overline{G} = 0$, so $i^n \langle \sigma, \sigma \rangle \lambda = \sigma \wedge \overline{\sigma}$ gives $\langle \sigma, \sigma \rangle = 2^{n+2} |x|^{n-2}$ and so $\langle \mu_G, \mu_G \rangle = 2^{n/2+1} |x|^{n/2-1}$. Thus

$$\|\Psi\|_G^2 = \int_X |\Psi|^2 \exp -4\pi|x| \; 2^{n/2+1} |x|^{n/2-1} \; |\lambda| .$$

$\mathcal{H}_G$ is then all holomorphic functions Ψ on X with $\|\Psi\|_G$ finite. The exponential convergence factor means $\mathcal{H}_G$ contains all polynomials in $z_0, \dots, z_n$ so is not trivial.

To pair F and G we need to compute $\pi^* \rho_0 \wedge \overline{\sigma}$. This is easily done on U_j using formulas (4) and (5) and the following expression

$$\lambda | U_j = 2 e_j^{-2} de_0 \wedge \cdots \wedge \widehat{de_j} \wedge \cdots \wedge de_n \wedge dx_0 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n .$$

One finds

$$\pi^* \rho_0 \wedge \overline{\sigma} = 2(-1)^n |x|^{-1} \lambda .$$

Thus

$$\langle \mu_F, \mu_G \rangle = i^n 2^{1/2} |x|^{-1/2} .$$

We shall drop the factor i^n since it makes no difference to the existence or unitarity. Denote the pairing of $\varphi s_F \otimes \mu_F$ and $\Psi s_G \otimes \mu_G$ by $\langle \varphi, \Psi \rangle$, then

$$\langle \varphi, \Psi \rangle = \int_X \varphi \overline{\Psi} \exp -2\pi|x| \; 2^{1/2} |x|^{-1/2} |\lambda| .$$

As a formal map $T : \mathcal{H}_G \longrightarrow \mathcal{H}_F$ the pairing can be written

$$(T\Psi)(e) = 2^{1/2} \int_{x \cdot e = 0} \Psi(|x|e + ix) \exp -2\pi|x| \; |x|^{-1/2} d^n x$$

with $d^n x$ the normalized Lebesgue measure on the cotangent space $\pi^{-1}(e)$.

§4. Existence and non-unitary nature of the pairing.

The proof of the existence of the pairing is based on being able to write down a kernel $K(z,e)$ analogous to that of §2. If $x \in \mathbb{R}^{n+1}$, $z \in \mathbb{C}^{n+1}$ and Δ_x denotes the Laplacian in the x-variables then

$$\Delta_x (x\cdot z)^k = k(k-1)\, z\cdot z\,(x\cdot z)^{k-2},$$

from which it follows that if $z \in X$, $(x\cdot z)^k$ is a homogeneous harmonic polynomial and therefore its restriction to the unit sphere is a spherical harmonic. For x fixed, as a function of z, $(x\cdot z)^k$ is holomorphic and polynomial and thus in $\mathcal{H}_G$. The spherical harmonics are dense in $\mathcal{H}_F$ and it will be shown that the polynomials in z are dense in $\mathcal{H}_G$. These will provide dense domains for T and T^{-1}.

Let $\mathcal{S}_k$ denote the spherical harmonics of order k on S^n and $\mathcal{P}_k$ the polynomials homogeneous of degree k on X. Then $\dim \mathcal{S}_k = \dim \mathcal{P}_k = (2k+n-1)\,\Gamma(k+n-1)/\{\Gamma(n)\,\Gamma(k+1)\}$ (this equality of dimension could be derived from our analysis of the relationship between $\mathcal{S}_k$ and $\mathcal{P}_k$ by working a little harder). Our first objective is to show T maps $\mathcal{P}_k$ isomorphically onto $\mathcal{S}_k$: this is the Laplace representation of elements of $\mathcal{S}_k$.

Define, for $\varphi \in \mathcal{S}_k$, $A_k\varphi \in \mathcal{P}_k$ by

$$(A_k\varphi)(z) = \int_{S^n} \varphi(a)\,(a\cdot z)^k\, |\rho_0|(da),$$

and for $\psi \in \mathcal{P}_k$ define $B_k\psi \in C(S^n)$ by

$$(B_k\psi)(a) = 2^{n/2+1}\int_X (a\cdot\bar z)^k \psi(z)\, \exp -4\pi|x|\; |x|^{n/2-1}\,|\lambda|.$$

The exponential convergence of the integrand justifies all the following manipulation.

$$(B_k\cdot A_k\varphi)(a) = 2^{n/2+1}\int_X (a\cdot\bar z)^k \int_{S^n}\varphi(b)(b\cdot z)^k\,|\rho_0|(db)\,\exp -4\pi|x|\;|x|^{n/2-1}\,|\lambda|$$

$$= \int_{S^n}\varphi(b)\,F(a,b)\,|\rho_0|(db),$$

where

$$F(a,b) = 2^{n/2+1}\int_X (a\cdot\bar z)^k (b\cdot z)^k \exp -4\pi|x|\;|x|^{n/2-1}\,|\lambda|.$$

$F(a,b)$ is a kernel defining a map of $\mathcal{S}_k$ to itself and clearly is $O(n+1)$ invariant. But $\mathcal{S}_k$ is an irreducible representation of $O(n+1)$, so $B_k\cdot A_k$ must be a multiple a_k of the identity. To find a_k we set $a=b$ and integrate

$$a_k\cdot \dim \mathcal{S}_k \cdot \operatorname{vol} S^n = 2^{n/2+1}\int_{S^n} F(a,a)\,|\rho_0|(da).$$

But $O(n+1)$ is transitive on S^n so $F(a,a)$ is constant. We can set $a = \epsilon_{n+1}$, the $(n+1)$th coordinate direction in $\mathbb{R}^{n+1}$ and then

$$a_k \dim \mathcal{S}_k = 2^{n/2+1}\int_X |z_{n+1}|^{2k} \exp -4\pi|x|\;|x|^{n/2-1}\,|\lambda|.$$

This integral is evaluated in the appendix to give

$$a_k = 2^{-4k-3n+5}\,\pi^{2k-n/2+3/2}\,\frac{\Gamma(2k+3n/2-1)\,\Gamma(n)\,\Gamma(k+1)^2}{(2k+n-1)\,\Gamma(n/2)\,\Gamma(k+(n+1)/2)\,\Gamma(k+n-1)}.$$

This is non-zero so A_k and B_k are invertible.

$\mathcal{H}_G$ is a unitary representation of $O(n+1)$ and by the above, $O(n+1)$ acts irreducibly on $\mathcal{P}_k$, so $\bigoplus_{k=0}^{\infty}\mathcal{P}_k$ is an orthogonal direct sum within $\mathcal{H}_G$. But in Lemma 1 of [9] I showed an holomorphic function f on X had an expansion $f=\sum_{k=0}^{\infty}f_k$ with $f_k\in\mathcal{P}_k$ so that $\mathcal{H}_G=\bigoplus_{k=0}^{\infty}\mathcal{P}_k$. Let $\mathcal{P}=\coprod_{k=0}^{\infty}\mathcal{P}_k$ be the algebraic sum. This is thus a dense domain in $\mathcal{H}_G$.

Let $\psi_1,\psi_2\in\mathcal{P}_k$ then $A_k:\mathcal{S}_k\longrightarrow\mathcal{P}_k$ is onto so $\psi_i=A_k\varphi_i$ with $\varphi_i\in\mathcal{S}_k$, $i=1,2$. Then

$$(\psi_1,\psi_2)_G = \int_X (A_k\varphi_1)(z)\,\overline{(A_k\varphi_2)(z)}\,\exp -4\pi|x|\;2^{n/2+1}|x|^{n/2-1}|\lambda|$$

$$= \int_{S^n}\int_{S^n}\varphi_1(a)\,\overline{\varphi_2(b)}\,F(a,b)\,|\rho_0|(da)\,|\rho_0|(db)$$

by a simple rearrangement. Thus

$$(\psi_1,\psi_2)_G = a_k\,(\varphi_1,\varphi_2)_F$$

Hence $a_k^{-1/2}A_k$ is unitary.

Now consider $T_k=T|\mathcal{P}_k$, and

$$(T_k\cdot A_k\varphi)(e) = 2^{1/2}\int_{x\cdot e=0}(A_k\varphi)(|x|e+ix)\,\exp -2\pi|x|\;|x|^{-1/2}\,d^nx$$

$$= \int_{S^n}\varphi(a)\,G(a,e)\,|\rho_0|(da)$$

with

$$G(a,b) = 2^{1/2}\int_{x\cdot b=0}\{a\cdot(|x|b+ix)\}^k\exp -2\pi|x|\;|x|^{-1/2}\,d^nx.$$

Again, $G(a,b)$ is $O(n+1)$-invariant and hence a multiple, b_k, of the identity. b_k is found, as before, by setting $a=b$ and integrating:

$$b_k = 2^{-k-n+2}\,\pi^{-k-n/2+1/2}\,\frac{\Gamma(k+n-\tfrac12)\,\Gamma(n)\,\Gamma(k+1)}{(2k+n-1)\,\Gamma(n/2)\,\Gamma(k+n-1)}.$$

Thus $T_k=b_ka_k^{-1}B_k$ and is $b_ka_k^{-1/2}$ times a unitary operator from $\mathcal{P}_k$ to $\mathcal{S}_k$. Also $\|T\|=\sup_k b_ka_k^{-1/2}$, $\|T^{-1}\|=\sup_k b_k^{-1}a_k^{1/2}$, if these exist.

We calculate $b_k^2a_k^{-1}$ as

$$\frac{2^{n/2}\,\Gamma(k+n-\tfrac12)^2\,\Gamma(k+(n-1)/2)}{\mathrm{vol}\,S^n\;\Gamma(k+n-1)\,\Gamma(k+3n/4)\,\Gamma(k+3n/4-\tfrac12)}.$$

This is monotone decreasing so $\|T\|=b_0a_0^{-1/2}$ is finite, and $\|T^{-1}\|=\lim_{k\to\infty}b_k^{-1}a_k^{1/2}$. But

$$\frac{\Gamma(k+\alpha_1)\;\ldots\;\Gamma(k+\alpha_r)}{\Gamma(k+\beta_1)\;\ldots\;\Gamma(k+\beta_r)}$$

has the limit ∞, 1 or 0 as $k\to\infty$ according as $\sum_{i=1}^{r}\alpha_i$ is greater, equal to or less than $\sum_{i=1}^{r}\beta_i$. In our case $\alpha_1=\alpha_2=n-\tfrac12$, $\alpha_3=(n-1)/2$, $\beta_1=n-1$, $\beta_2=3n/4$, $\beta_3=3n/4-\tfrac12$ so $\alpha_1+\alpha_2+\alpha_3=5n/2-3/2=\beta_1+\beta_2+\beta_3$, so that $\|T^{-1}\|=(\mathrm{vol}\,S^n)^{1/2}\,2^{-n/4}$, which is finite. Thus T and T^{-1} are bounded and hence we have established the rigorous existence of the pairing. Since $\|T_k\|$ is properly decreasing, T is not unitary, nor a multiple of a unitary operator.

The flow σ_t preserves G, so lifts into L and satisfies $\sigma_t\cdot S_G=S_G$. Also one finds from (5) that

$$\sigma_t^*\,\sigma = \exp\{-(n-1)2\pi it\}\,\sigma$$

and so

$$\sigma_t^*\,\mu_G = \exp\{-(n-1)\pi it\}\,\mu_G.$$

Thus σ_t quantizes on $\mathcal{H}_G$ to give the unitary group U_t with

$$(U_t\psi)(z) = \exp(n-1)\pi i t \ \psi(\exp 2\pi i t\, z).$$

For $\psi \in \mathcal{P}_k$ we have

$$U_t\psi = \exp\{(k+(n-1)/2)2\pi i t\}\ \psi,$$

so that $TU_tT^{-1} = \exp\{2\pi i t\,[-\Delta+(n-1)^2/4]^{1/2}\}$, as the latter group has the same spectrum and eigenspaces.

Appendix.

To evaluate

$$C_k = \int_X |z_{n+1}|^2 \exp -4\pi|x|\ |x|^{n/2-1}\,|\lambda|,$$

write $x = ry$ with $y\cdot y = 1$, $y\cdot e = 0$ then

$$C_k = \int_0^\infty r^{2k+3n/2} \exp -4\pi r\, dr \int_{\substack{y\cdot e=0\\ y\cdot y=e\cdot e=1}} (e_{n+1}^2 + y_{n+1}^2)^k\, d\mathrm{vol},$$

$$= (4\pi)^{-2k-3n/2+1}\,\Gamma(2k+3n/2-1)\, I_k$$

where

$$I_k = \int_{\substack{y\cdot e=0\\ y\cdot y=e\cdot e=1}} (a\cdot e^2 + a\cdot y^2)^k\, d\mathrm{vol}.$$

This is independent of a, so integrating over a gives

$$I_k \,\mathrm{vol}\, S^n = \int_{S^n} \int_{\substack{y\cdot e=0\\ e\cdot e=y\cdot y=1}} (a\cdot e^2 + a\cdot y^2)^k\, d\mathrm{vol}\ |\rho_0|(da)$$

$$= \int_{\substack{y\cdot e=0\\ e\cdot e=y\cdot y=1}} \int_{S^n} (a\cdot e^2 + a\cdot y^2)^k\, |\rho_0|(da)\, d\mathrm{vol}.$$

But $O(n+1)$ is transitive on the set of pairs (e,y), $e\cdot e = y\cdot y = 1$, $e\cdot y = 0$, so

$$\int_{S^n} (a\cdot e^2 + a\cdot y^2)^k\, |\rho_0|(da)$$

is independent of (e,y). We can therefore evaluate it by setting $e = \epsilon_{n+1}$, $y = \epsilon_n$. Then

$$I_k \,\mathrm{vol}\, S^n = \mathrm{vol}\, S^n\, \mathrm{vol}\, S^{n-1} \int_{S^n} (e_{n+1}^2 + e_n^2)^k\, |\rho_0|(da).$$

This last integral we write in spherical polar coordinates :

$$I_k = \mathrm{vol}\,S^{n-1}\mathrm{vol}\,S^{n-2}\int_0^\pi\int_0^\pi (\cos^2\theta + \sin^2\theta\cos^2\varphi)^k \sin^{n-1}\theta\sin^{n-2}\varphi\, d\theta d\varphi$$

$$= \mathrm{vol}\,S^{n-1}\mathrm{vol}\,S^{n-2}\int_0^\pi\int_0^\pi (1-\sin^2\theta\sin^2\varphi)^k \sin^{n-1}\theta\sin^{n-2}\varphi\, d\theta d\varphi$$

$$= \mathrm{vol}\,S^{n-1}\mathrm{vol}\,S^{n-2}\sum_{r=0}^{k} (-1)^r \binom{k}{r} J_{2r+n-1} J_{2r+n-2}$$

where

$$J_k = \int_0^\pi \sin^k\theta\, d\theta$$

Now, integrating by parts,

$$J_k = -\sin^{k-1}\theta\cos\theta\Big|_0^\pi + (k-1)\int_0^\pi \sin^{k-2}\theta\cos^2\theta\, d\theta$$

$$= (k-1)(J_{k-2} - J_k)$$

for $k \geqslant 2$. Thus

$$k J_k = (k-1) J_{k-2}$$

for $k \geqslant 2$. Multiplying both sides by J_{k-1}, we see $k J_k J_{k-1}$ is constant, so

$$k J_k J_{k-1} = J_1 J_0 = 2\pi .$$

Thus

$$I_k = \mathrm{vol}\,S^{n-1}\mathrm{vol}\,S^{n-2}\sum_{r=0}^{k} (-1)^r \binom{k}{r} \frac{2\pi}{2r+n-1}$$

$$= 2\pi\,\mathrm{vol}\,S^{n-1}\mathrm{vol}\,S^{n-2}\int_0^1 x^{n-2}(1-x^2)^k\, dx .$$

Another integration by parts procedure shows that the last integral is

$$\tfrac{1}{2}\Gamma(k+1)\Gamma((n-1)/2)\,/\,\Gamma(k+(n+1)/2).$$

Then

$$I_k = 4\pi^{n+\frac{1}{2}}\Gamma(k+1)\,/\,\{\Gamma(n/2)\Gamma(k+(n+1)/2)\}.$$

References.

1. V. Bargmann. On a Hilbert space of analytic functions and an associated integral transform. Comm. Pure Appl. Math. 20 (1967) 1-101.

2. R. Blattner. The metalinear geometry of non-real polarizations. Lecture Notes in Mathematics 570 Springer-Verlag, Berlin-Heidelberg-New York, 1977.

3. J. Elhadad. Quantification du flot géodésique de la sphère S^n. CNRS, Marseille, June 1977.

4. B. Kostant. Quantization and unitary representations I. Lecture Notes in Mathematics 170. Springer-Verlag, Berlin-Heidelberg-New York, 1970.

5. B. Kostant. Symplectic spinors. Symposia Mathematica XIV. Academic Press, London, 1974.

6. B. Kostant. On the definition of quantization. Colloque symplectique, Aix-en-Provence. CNRS 1974.

7. F. J. Moser. Regularization of Kepler's problem and the averaging method on a manifold. Comm. Pure Appl. Math. 23 (1970) 609-636.

8. C. Müller. Spherical harmonics. Lecture Notes in Mathematics 17, Springer-Verlag, Berlin-Heidelberg-New York 1966.

9. L. Nirenberg. A complex Frobenius theorem. Seminars on analytic functions I. Princeton Univ. Press. Princeton, 1957, pp. 172-189.

10. J. Rawnsley. Coherent states and Kaehler manifolds. Quarterly J. Math. (1977, to appear).

11. J. Rawnsley. On the pairing of polarizations. Comm. Math. Phys. (1978, to appear).

12. J. Rawnsley. On the cohomology groups of a polarization and diagonal quantization. Trans. Amer. Math. Soc. 230 (1977) 235-255.

13. D. Simms, N. Woodhouse. Lectures on geometric quantization. Lecture Notes in Physics 53. Springer-Verlag, Berlin-Heidelberg-New York, 1976.

14. J.-M. Souriau. Structure des systèmes dynamiques. Dunod, Paris, 1970.

15. J.-M. Souriau. Sur la variété de Kepler. Symposia Mathematica XIV, Academic Press, London, 1974.

16. A. Weinstein. Quasi-classical mechanics on spheres. Symposia Mathematica XIV, Academic Press, London, 1974.