DIAS-TP-78-04

Flat Partial Connections and Holomorphic Structures in C^∞ Vector Bundles

J. H. Rawnsley

School of Theoretical Physics

Dublin Institute for Advanced Studies

Dublin 4, Ireland

Abstract

The notion of a flat partial connection D in a C^∞ vector bundle E, defined on an integrable sub-bundle F of the complexified tangent bundle of a manifold X is defined. It is shown that E can be trivialized by local sections s satisfying Ds = 0. The sheaf of germs of sections s of E satisfying Ds = 0 has a natural fine resolution, giving the de Rham and Dolbeault resolutions as special cases.

If X is a complex manifold and F the tangents of type (0, 1), the flat partial connections in a C^∞ vector bundle E are put in correspondence with the holomorphic structures in E.

If X, E are homogeneous and F invariant, then invariant flat connections in E can be characterized as extensions of the representation of the isotropic subgroup to which E is associated, extending results of Tirao and Wolf in the holomorphic case.

1. Introduction

Let E be a holomorphic vector bundle over a complex manifold X and $TX^c = F \oplus \overline{F}$ the splitting of the tangent bundle of X into subbundles of types $(0,1)$ and $(1,0)$ respectively. Then F is closed under Lie bracket, and there is a unique first order differential operator

$$D : \Gamma E \longrightarrow \Gamma F^* \otimes E \tag{1}$$

such that

$$D(fs) = f Ds + \bar{\partial} f \otimes s \tag{2}$$

for f in $C^\infty(X)$, s in ΓE and where $Ds = 0$ on an open set U if and only if s is holomorphic on U. If we put

$$\nabla_\xi s = (Ds)(\xi) \quad , \quad \xi \in \Gamma F, \tag{3}$$

identifying $F^* \otimes E$ with $Hom(F, E)$, then ∇_ξ behaves like a covariant derivative in E, but is only defined for ξ in ΓF. Moreover ∇ is flat:

$$\nabla_{[\xi,\eta]} = \nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi \quad , \quad \xi, \eta \in \Gamma F. \tag{4}$$

If we begin with a C^∞ vector bundle E over X and a differential operator D as in (1), satisfying (2), we can ask if E always has a holomorphic structure such that the solutions of $Ds = 0$ are precisely the (local) holomorphic sections of E. The answer is yes, provided the operators ∇ defined by (3) satisfy (4). This is the Corollary to the Theorem (see below). It is useful to encode the holomorphic structure as such an operator D or ∇, since operations on the category of C^∞ vector bundles often extend automatically to the category of C^∞ vector bundles with connections (or partial connections).

For example, if K and Q are line bundles with $i: Q^2 \longrightarrow K$ an isomorphism and ∇^K a partial connection in K, there is a unique partial connection ∇^Q in Q such that

$$\nabla^K_\xi i(s_1 \otimes s_2) = i(\nabla^Q_\xi s_1 \otimes s_2 + s_1 \otimes \nabla^Q_\xi s_2)$$

for all ξ for which ∇^K is defined and all sections s_1, s_2 of Q. Then ∇^Q is flat if and only if ∇^K is. If K is holomorphic and ∇^K the partial connection on the $(0,1)$ tangent bundle defined above, ∇^K is flat and hence Q has a flat partial connection on the $(0,1)$ tangents. Then Q has a holomorphic structure and the isomorphism $i: Q^2 \longrightarrow K$ becomes an isomorphism of holomorphic line bundles. cf. [5].

Partial connections were introduced by Bott [2] for foliations. We can combine the real foliation, and complex structure versions as follows: If X is any manifold, $TX^{\mathbb{C}}$ the complexified tangent bundle, a subbundle $F \subset TX^{\mathbb{C}}$ is integrable if

(i) $F \cap \overline{F}$ has constant rank,

(ii) F and $F + \overline{F}$ are closed under Lie bracket.

Then according to Nirenberg [7], X can be covered by open sets U on which there are coordinates $u_1, \dots, u_k, v_1, \dots, v_\ell, x_1, \dots, x_m, y_1, \dots, y_m$ where, if $z_j = x_j + \sqrt{-1}\, y_j$, $j = 1, \dots, m$, F is spanned on U by

$$\partial/\partial u_1, \dots, \partial/\partial u_k, \partial/\partial \bar{z}_1, \dots, \partial/\partial \bar{z}_m .$$

If f is a C^∞ function on X, let $d^F f$ denote the restriction of df to F, regarded as a section of the dual bundle F^*. Putting $\Omega^p_F = \Gamma \Lambda^p F^*$, d^F extends to a differential

$$d^F : \Omega^p_F \longrightarrow \Omega^{p+1}_F , \quad p \geq 0,$$

with all the usual properties, including a local Poincaré Lemma (see [8]).

Let E be a C^∞ vector bundle over X, then a partial connection defined on F, or an F-connection is a linear map

$$D : \Gamma E \longrightarrow \Gamma F^* \otimes E$$

satisfying

$$D(fs) = f\, Ds + d^F f \otimes s$$

for all f in $C^\infty(X)$, s in ΓE. D extends to a map

$$D : \Omega^p_F(E) \longrightarrow \Omega^{p+1}_F(E) , \quad p \geq 0,$$

where $\Omega^p_F(E) = \Gamma \Lambda^p F^* \otimes E$. $D \circ D$ defines a section R of $\Lambda^2 F^* \otimes \mathrm{End}(E)$ which is the curvature, and we say D is flat if $R = 0$.

An example of a flat F-connection may be obtained by generalizing Bott's construction. Let $F^\circ \subset T^* X^{\mathbb{C}}$ be all covectors vanishing on F. We define

$$D : \Gamma F^\circ \longrightarrow \Gamma F^* \otimes F^\circ$$

by

$$(Ds)(\xi) = \xi \lrcorner ds , \quad s \in \Gamma F^\circ, \xi \in \Gamma F. \tag{5}$$

This makes sense since s is a 1-form on X. Since $\xi \lrcorner s = 0$ for all ξ in ΓF, s in ΓF°, the right hand side of (5) is the Lie derivative of s with respect to ξ. This shows D is flat. In the case $F = \overline{F}$, F is the tangent bundle to a foliation, F° the (co-) normal bundle and D is Bott's connection along the leaves of F.

Theorem 1. A C^∞ vector bundle E admits a flat F-connection D, where F is integrable, if and only if it can be trivialized (locally) by sections s satisfying $Ds = 0$.

Corollary. Let X be a complex manifold, F the bundle of tangents of type $(0,1)$ and E a C^∞ vector bundle with a flat F-connection D then E has a unique holomorphic structure such that the holomorphic sections on an open set U are the solutions of $Ds = 0$ on U.

The corollary is an immediate consequence of theorem 1. Theorem 1 is proven in §2, and further applications in §3. Operators such as D are examples of overdetermined systems considered in [4]. In the case at hand a simple direct proof of theorem 1 can be given using estimates from [6], its only being necessary to check that these estimates imply smooth dependence on parameters.

A version of these results for line bundles already appears in [8] with applications to Kostant's theory of geometric quantization.

N. J. Hitchin, in joint work with M. F. Atiyah and I. M. Singer, has an alternative proof of the corollary [1], and I would like to thank him for several useful conversations on this topic. I would also like to thank J. T. Lewis for his valuable help and advice in the preparation of this paper.

2. Proof of theorem 1.

$F \cap \overline{F}$ is real and integrable. We can choose, through any given point x, a submanifold Y of X transversal to the leaves of $F \cap \overline{F}$. Then $F' = F|Y$ satisfies $F' \cap \overline{F'} = 0$ and is integrable. If we solve the problem on Y we can parallelly translate the sections along the leaves of $F \cap \overline{F}$ and so obtain a solution in a neighbourhood of x. Thus we may assume $F \cap \overline{F} = 0$.

If $F \cap \overline{F} = 0$ we have coordinates $v_1, \ldots, v_\ell, x_1, \ldots x_m, y_1, \ldots, y_m$, with F spanned by

$$\partial/\partial \bar z_1, \ldots, \partial/\partial \bar z_m$$

where $z_j = x_j + \sqrt{-1}\, y_j$, $j = 1, \ldots, m$, as before. Choose a local frame field $s_1, \ldots, s_N$ for E on this coordinate neighbourhood and define matrices A_j of functions by

$$\nabla_{\partial/\partial \bar z_j} s_b = \sum_{a=1}^{N} (A_j)_{ab} s_a, \quad b = 1, \ldots, N, j = 1, \ldots, m.$$

Then ∇ is flat if

$$\partial A_j/\partial \bar z_i - \partial A_i/\partial \bar z_j + [A_i, A_j] = 0, \quad i, j = 1, \ldots, m \tag{6}$$

Put $A = \sum_{j=1}^{m} A_j d\bar z_j$ and regard $v_1, \ldots, v_\ell$ as parameters, then equations (6) become

$$\bar\partial A + A \wedge A = 0. \tag{7}$$

This is the formal integrability condition for having a matrix g of functions which is invertible and satisfying

$$\bar\partial g + A g = 0. \tag{8}$$

It is shown in [6] that, when there are no parameters, (8) always has a solution provided (7) holds. We shall check that the proof of [6] goes through with smooth dependence on parameters so that g is a C^∞ function of $v_1, \dots, v_\ell$, $x_1, \dots, x_m, y_1, \dots, y_m$. Then we may define

$$t_b = \sum_{a=1}^{N} g_{ab} s_a , \qquad b = 1, \dots, N$$

and obtain a frame field $t_1, \dots, t_N$ satisfying

$$Dt_a = 0 , \qquad a = 1, \dots, N.$$

This will complete the proof of the theorem.

The proof in [6] uses an explicit homotopy operator T for $\bar\partial$ in a polycylinder of radius R in the coordinates $z_1, \dots, z_m$:

$$\beta = T\bar\partial\beta + \bar\partial T\beta$$

for every (p,q)-form β with $q > 0$. A Hölder norm $\|\cdot\|$ is defined on forms on this polycylinder and it is shown that

$$\|T\| \leqslant c_1 R$$

for some constant $c_1 > 0$. Moreover, $\|A\|$ depends continuously on $v_1, \dots, v_\ell$ (as parameters) and, restricting them to a fixed compact neighbourhood, we have

$$\|A\| \leqslant c_2$$

uniformly in $v_1, \dots, v_\ell$.

Thus the operator $f \longmapsto T(Af)$ on matrices of functions satisfies

$$\|T(Af)\| \leqslant c_1 c_2 R \|f\|,$$

and by choosing R so that $c_1 c_2 R < 1$, a contraction mapping is obtained.

Then if g is a solution of

$$g = \psi - T(Ag) \tag{9}$$

where $\bar\partial\psi = 0$, we have

$$\begin{aligned}\bar\partial g &= -\bar\partial T(Ag) = -Ag + T(\bar\partial(Ag)) \\ &= -Ag + T((\bar\partial A)g) - T(A \wedge \bar\partial g) \\ &= -Ag - T(A \wedge (\bar\partial g + Ag)).\end{aligned}$$

Thus

$$\bar\partial g + Ag = T(A \wedge (\bar\partial g + Ag))$$

and, since $T \circ A$ is a contraction, we must have

$$\bar\partial g + Ag = 0.$$

Moreover, by the uniqueness of fixed points, g depends on $v_1, \dots, v_\ell$. By choosing ψ suitably, and modifying T as in [6] we can make sure g is invertible at x, and hence in a neighbourhood of x (when we have established continuity).

g depends differentiably on $x_1, \dots, x_m, y_1, \dots, y_m$ if A does, from the proof in [6], when $v_1, \dots, v_\ell$ are held fixed. To see that it also depends differentiably on $v_1, \dots, v_\ell$, we observe that formally differentiating (9) gives

$$\partial g/\partial v_i = T(\partial A/\partial v_i \, g) - T(A \, \partial g/\partial v_i), \tag{10}$$

an equation of the same kind as (9).

Solutions to both equations (9) and (10) are obtained iteratively by setting

$$g_{n+1} = \psi + T(Ag_n)$$

and $g = \lim_{n\to\infty} g_n$. If formal differentiation inside T is allowed, the sequence $\partial g_n/\partial v_i$ converges for each i , uniformly in $v_1,\dots,v_\ell$ and so the limit exists and is continuous by standard results in analysis. T is built from integral operators acting on the variables $z_1,\dots,z_m$ one by one. It suffices, therefore, to consider the case $m = 1$. We abbreviate $v_1,\dots,v_\ell$ by v . The operators are of the form

$$(Kf)(v,z) = \oint_{|\zeta|=R} f(v,\zeta)/(\zeta-z)\, d\zeta, \quad (Lf)(v,z) = \iint_{|\zeta|\le R} f(v,\zeta)/(\zeta-z)\, d\zeta\wedge d\bar\zeta.$$

The first clearly causes no difficulties when $|z| < R$. In the second, the methods of [3, p.21] allow the integrand to be improved to

$$(L'f)(v,z) = \iint_{|\zeta|\le R} f(v,\zeta)\log|\zeta-z|^2\, d\zeta\wedge d\bar\zeta.$$

But $\log|\zeta-z|$ is the fundamental solution of the Laplacian in the plane, and standard results from potential theory, see for example [9], imply that $L'f$ is C^∞ if f is. Thus T maps C^∞ forms to C^∞ forms. This concludes the proof.

3. Applications.

Let E be a C^∞ vector bundle over X , $F \subset TX^{\mathbb{C}}$ an integrable subbundle and D a flat F-connection. Define $\Omega^p_F(E)$ as in the introduction. Let $\mathcal{A}^p$ denote the sheaf associated to the presheaf $U \mapsto \Omega^p_F(E|U)$ and D the induced map from $\mathcal{A}^p$ to $\mathcal{A}^{p+1}$. Let $\mathcal{S}$ be the sheaf of germs of solutions of $Ds = 0$ which is a subsheaf of $\mathcal{A}^0$, then we have a sequence

$$0 \to \mathcal{S} \hookrightarrow \mathcal{A}^0 \xrightarrow{D} \mathcal{A}^1 \xrightarrow{D} \mathcal{A}^2 \to \dots\dots \qquad (11)$$

Theorem 2. (11) is a fine resolution of $\mathcal{S}$ so that the cohomology groups $H^p(\mathcal{S})$ are isomorphic to those of the complex

$$\Omega^0_F(E) \xrightarrow{D} \Omega^1_F(E) \xrightarrow{D} \Omega^2_F(E) \to \dots$$

Proof. The sheaves $\mathcal{A}^p$ are clearly fine, and $D\circ D = 0$ since D is flat. It remains to prove that if $D\beta = 0$, β in $\Gamma(\mathcal{A}^p, U), p>0$ for some open set U then for each x in U there is an open set V in U containing x and α in $\Gamma(\mathcal{A}^{p-1}, V)$ with $\beta|V = D\alpha$. But by theorem 1 there is an open set W in U containing x which has a local frame field $s_1,\dots,s_N$ for E with $Ds_i = 0$. Then on W ,

$$\beta = \sum_{a=1}^{N} \beta_a \otimes s_a$$

with $\beta_a \in \Omega^p_{F|W}$, and $D\beta = 0$ implies $d^F\beta_a = 0$. But $\beta_a = d^F\alpha_a$ on some common niehgbourhood V of x , by the Poincaré lemma for d^F and then $\alpha = \sum_{a=1}^{N} \alpha_a \otimes s_a$ gives the required section.

Remark. Theorem 2 contains the usual de Rham and Dolbeault isomorphisms as special cases by taking F real or $TX^{\mathbb{C}} = F \oplus \bar F$.

A second application generalizes some of the results of [10]. Let

$X = G/H$ be a homogeneous space for a Lie group G, and let $F \subset TX^{\mathbb{C}}$ be invariant. Then there is a subspace $\mathfrak{p} \subset \mathfrak{g}^{\mathbb{C}}$ ($\mathfrak{g}$ the Lie algebra of G) containing $\mathfrak{h}$ and $Ad_G H$ -stable, which, when translated around X from the identity coset, gives F. F is closed under Lie bracket if and only if $\mathfrak{p}$ is a subalgebra, and F is integrable if, in addition, $\mathfrak{p} + \bar{\mathfrak{p}}$ is a subalgebra. Let E be a homogeneous vector bundle over X and $g \cdot s$, g in G , s in ΓE the induced action of G on sections of E. Let $g \cdot \xi$ denote the induced action on sections of F, then an F-connection D in E is invariant if

$$g \cdot (\nabla_{\xi} s) = \nabla_{g \cdot \xi}\, g \cdot s$$

for ξ in ΓF , s in ΓE and g in G.

If we differentiate the action of G on ΓE we get a representation of $\mathfrak{g}$ and we extend it to $\mathfrak{g}^{\mathbb{C}}$ by linearity. For $a \in \mathfrak{g}^{\mathbb{C}}$ let ξ^a be the vector field it determines on X so that F_{eH} is generated by $\{\xi^a_{eH} \mid a \in \mathfrak{p}\}$. Then for $a \in \mathfrak{p}$ we have two operations on ΓE , namely $a \cdot s$ and $\nabla_{\xi^a} s$. Moreover

$$a \cdot (fs) = f\, a \cdot s - \xi^a(f)\, s$$

for f in $C^\infty(X)$, s in ΓE and a in $\mathfrak{g}^{\mathbb{C}}$. Thus

$$a \cdot s + \nabla_{\xi^a} s \quad , \quad a \in \mathfrak{p}$$

is $C^\infty(X)$ -linear in s and hence by evolution at the identity coset eH defines an endomorphism $\gamma(a)$ of E_{eH}. For a in $\mathfrak{h}$ this is the derivative of the action of H on E_{eH} defining E, since $\xi^a = 0$.

Proposition. γ determines D uniquely, and D is flat if and only if γ is a representation of $\mathfrak{p}$ on E_{eH}.

The details of the proof are straightforward and are left to the reader. [10] dealt with the case where X was complex and E holomorphic. In the Kostant-Kirillov-Dixmier programme for constructing representations, subalgebras $\mathfrak{p}$ (polarizations) arise which do not correspond with complex structures. Further applications of the notion of flat partial connections will appear elsewhere.

References.

1. Atiyah, M. F., Hitchin, N.J., Singer, I. M. Self-duality in four-dimensional Riemannian geometry. Oxford preprint, 1977.

2. Bott, R. Lectures on characteristic classes and foliations. Lecture Notes in Mathematics 279, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

3. Chern, S. S. Complex manifolds without potential theory. Mathematical Studies No. 15, Van Nostrand, Princeton, N.J., 1967.

4. Goldschmidt, H., Spencer, D. Submanifolds and over-determined differential operators. Complex analysis and algebraic geometry, ed. W. L. Bailey and T. Shioda. Cambridge University Press, Cambridge, 1977.

5. Hitchin, N. J. Harmonic spinors. Advances in Math. 14 (1974), 1-55.

6. Nijenhuis, A., Woolf, W. Some integration problems in almost-complex and complex manifolds. Annals of Math. 77 (1963), 424-489.

7. Nirenberg, L. A complex Frobenius theorem. Seminars on analytic functions Princeton University Press, Princeton, N.J., 1957, pp. 172-189.

8. Rawnsley, J. H. On the cohomology groups of a polarization and diagonal quantization. Trans. Amer. Math. Soc. 230 (1977), 235-255.

9. Sternberg, W., Smith, T. The theory of potential and spherical harmonics. University of Toronto Press, Toronto, 1961.

10. Tirao, J. A., Wolf, J. A. Homogeneous holomorphic vector bundles. Indiana Univ. Math. J. 20 (1970), 15-31.