Multipseudoparticles and Perturbation Theory in Large Order of the Anharmonic Oscillator

by

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Abstract

Multi-instantons are included in a calculation of the ground-state energy of the anharmonic oscillator in large orders of perturbation theory. Their inclusion verifies a conjecture of Langer and results in a partition function that reproduces the Bender-Wu result.

December 1977. To be published in Physical Review D.

The general formalism recently developed by Lipatov[1] for calculating large order perturbation theory in quantum field theories has been applied to the Hamiltonian

$$H = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\varphi^2 + \frac{g}{4}\varphi^4,$$

by several authors.[2,3] The object of interest is the large order behaviour of the coefficients in the perturbation expansion of its ground-state energy E(g), which takes the form

$$E(g) = \frac{1}{2} + \sum_{n=1}^{\infty} E_n g^n .$$

This calculation is of considerable importance since it is the only non-trivial one so far where the result may be directly compared with a previously obtained result, in this case, that of Bender and Wu.[4]

In the calculation of Brézin, Le Guillou and Zinn-Justin,[2] the possible contribution of multi-instantons to the leading behaviour of E_n is not discussed. They can be neglected if the interchange of the large n limit with the zero temperature limit in the derivation of their Eq. (32) is permissible. This interchange of limits is apparently valid but probably hard to prove. Given its validity, Eq. (32) follows because the multi-instantons have higher Euclidean action than the instanton given by their Eqs. (13) - (14) and therefore do not contribute to the dominant growth of the coefficient of g^n in the partition function.

The approach of Collins and Soper[3] is somewhat different than the authors in Ref. 2 as it is more closely connected with earlier ideas of Dyson[5] and Langer.[6] It is of interest here because the interchange of the large n limit with the zero temperature limit is avoided. However, their calculation should be completed by including multi-instantons, which we will do here. When this is done the result of Bender and Wu again follows.

The essentials of the calculation of Collins and Soper[3] can be stated as follows: For $n \geq 1$, E_n is given by[4]

$$E_n = \frac{(-1)^{n+1}}{\pi} \int_0^\infty \frac{d\lambda}{\lambda^{n+1}} \operatorname{Im} E(-\lambda). \tag{1}$$

The imaginary part of $E(-\lambda)$ may be obtained from

$$E(-\lambda) = \frac{1}{2} - \lim_{\beta \to \infty} \left(\frac{1}{\beta} \ln Z(-\lambda) \right), \tag{2}$$

with the partition function Z given by the Feynman-Kac integral over periodic paths of period β :

$$Z(-\lambda) = N \int_{x(\beta/2)=x(-\beta/2)} d[x] \exp\left[- \int_{-\beta/2}^{\beta/2} dt \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 - \frac{\lambda}{4}x^4 \right) \right].$$

The normalization factor N is chosen so that $Z(0) = 1$. It is assumed that the continuation of Z to negative coupling by replacing g by $\lambda e^{i\vartheta}$ and letting $\vartheta \to \pi$ can be performed, presumably by a suitable modification of the space of periodic paths on which the integral for Z is defined. Since the object of interest is E_n for large n, it is evident from (1) that an estimate of $Z(-\lambda)$ for small λ will suffice. An estimate of Z in this case can be obtained by summing over the leading saddle points in x, which are given by

$$\ddot{x} - x + \lambda x^3 = 0. \tag{3}$$

After $x(t) = 0$, the next lowest contribution to the Euclidean action

$$S[x] = \int_{-\beta/2}^{\beta/2} dt \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 - \frac{\lambda}{4}x^4 \right) \tag{4}$$

is[2,3,6]

$$x_\pm(t) = \pm \left(\frac{2}{\lambda} \right)^{1/2} \operatorname{sech}(t - \tau), \tag{5}$$

where τ is an arbitrary time origin and the plus (minus) sign denotes an (anti-) instanton. The periodicity condition $x_\pm(\beta/2) = x_\pm(-\beta/2)$ is satisfied in the limit $\beta \to \infty$. The inclusion of quadratic fluctuations about $x_\pm$ results in a contribution to Z given by

$$-i\beta \left(\frac{8}{\lambda\pi} \right)^{1/2} e^{-4/3\lambda}, \tag{6}$$

where the factor $4/3\lambda$ in the exponential is the value of $S[x_\pm]$. Adding to this the quadratic fluctuations about the saddle point at $x(t) = 0$ gives the following estimate for Z for small λ :

$$Z(-\lambda) = 1 - i\beta \left(\frac{8}{\lambda\pi} \right)^{1/2} e^{-4/3\lambda}. \tag{7}$$

This is one of the main results of the authors in ref. 3. It is also precisely the result of Langer.[6] We now wish to show that if multi-instantons are included in the calculation of Z the result (7) exponentiates, as conjectured by Langer.[6]

For purposes of illustration we consider two widely separated instantons. The case of n such instantons follows as a trivial generalization. Four quasi solutions of (3) are

$$x_{\pm\pm} = \pm \left(\frac{2}{\lambda} \right)^{1/2} \operatorname{sech}(t - \tau_1) \pm \left(\frac{2}{\lambda} \right)^{1/2} \operatorname{sech}(t - \tau_2),$$

with τ_1 and τ_2 satisfying the conditions in (8) below, but otherwise arbitrary. These are quasi solutions in the sense that the right-hand side of (3) is at most $O(e^{-|\tau_1 - \tau_2|})$ for $|t| \leq \beta/2$ provided

$$|\tau_1 - \tau_2| \gg 1, \qquad |\tau_1|, |\tau_2| \ll \beta. \tag{8}$$

Given (8) they also have the property that

$$S[x_{\pm\pm}] = 2S[x_\pm] + O(e^{-|\tau_1 - \tau_2|}),$$

where S and $X_{\pm}$ are given by (4) and (5). Two overlapping instantons are not stationary points of the action and only give a contribution to Z that grows as β instead of β^2 (see below). Hence we neglect them here.

Now consider fluctuations about an instanton-instanton pair,

$$x = x_{++} + y .$$

The contribution to Z from this (quasi) saddle point, up to terms quadratic in y, is obtained from

$$\tilde{Z}_{++} = e^{-8/3\lambda} N \int_{y(\beta/2)=y(-\beta/2)} d[y]\, e^{-\frac{1}{2}\langle y|H_{++}|y\rangle} , \tag{9}$$

where

$$H_{++} = -\partial_t^2 + 1 - 6\left(\operatorname{sech}(t-\tau_1) + \operatorname{sech}(t-\tau_2)\right)^2 . \tag{10}$$

For $|\tau_1 - \tau_2| \gg 1$, the cross term in (10) is negligible so that

$$H_{++} \simeq -\partial_t^2 + 1 - 6\sum_{i=1}^{2} \operatorname{sech}^2(t-\tau_i)$$
$$\simeq H_{+-} \simeq H_{-+} \simeq H_{--} .$$

The calculation of the functional integral in (9) requires the value of the Fredholm determinants

$$D = \frac{\det\left(-\partial_t^2 + 1 - 6\sum_{i=1}^{2}\operatorname{sech}^2(t-\tau_i)\right)}{\det(-\partial_t^2 + 1)} . \tag{11}$$

To evaluate (11) we follow the general procedure of Coleman.[7] Thus

$$D = \exp \operatorname{tr} \ln\left(1 - \frac{6}{-\partial_t^2+1} \sum_i \operatorname{sech}^2(t-\tau_i)\right)$$
$$= \exp \operatorname{tr}\left(\frac{-6}{-\partial_t^2+1} \sum_i \operatorname{sech}^2(t-\tau_i) - \frac{6^2}{2}\frac{1}{-\partial_t^2+1} \sum_i \operatorname{sech}^2(t-\tau_i)\right.$$
$$\left. \times \frac{1}{-\partial_t^2+1} \sum_j \operatorname{sech}^2(t-\tau_j) - \cdots \right) . \tag{12}$$

Since

$$\operatorname{sech}^2(t-\tau_i)\operatorname{sech}^2(t-\tau_j) \cdots \simeq 0 ,$$

for $i \neq j \neq \cdots$ when $|\tau_1 - \tau_2| \gg 1$, (12) can be rewritten as

$$D \simeq \exp \sum_{i=1}^{2} \operatorname{tr}\left(\frac{-6}{-\partial_t^2+1}\operatorname{sech}^2(t-\tau_i) - \frac{6^2}{2}\frac{1}{-\partial_t^2+1}\operatorname{sech}^2(t-\tau_i)\right.$$
$$\left. \times \frac{1}{-\partial_t^2+1}\operatorname{sech}^2(t-\tau_i) - \cdots\right)$$

$$= \frac{\prod_{i=1}^{2} \det\left(-\partial_t^2 + 1 - 6\operatorname{sech}^2(t-\tau_i)\right)}{\det^2(-\partial_t^2+1)} . \tag{13}$$

On the basis of (13)

$$N \int_{y(\beta/2)=y(-\beta/2)} d[y]\, e^{-\frac{1}{2}\langle y|H_{++}|y\rangle} \simeq N^2 \prod_{i=1}^{2} \int_{y(\beta/2)=y(-\beta/2)} d[y]\, e^{-\frac{1}{2}\langle y|H_i|y\rangle} , \tag{14}$$

where

$$H_i = -\partial_t^2 + 1 - 6\operatorname{sech}^2(t-\tau_i) . \tag{15}$$

Thus the calculation of quadratic fluctuations about a widely separated instanton-instanton pair reduces to the calculation of quadratic fluctuations about two isolated instantons.

The zero-frequency mode Ψ_0 of H in (15) can be projected out of the fluctuations about the two isolated instantons following the procedure of Javicki[3] and Collins and Soper.[3] Then (14) becomes

$$N \int_{y(\beta/2)=y(-\beta/2)} d[y]\, e^{-\frac{1}{2}\langle y|H_{++}|y\rangle} \simeq \int_{-\beta/2}^{\beta/2} d\tau_1 \int_{-\beta/2}^{\tau_1} d\tau_2 \Big[N \int_{y(\beta/2)=y(-\beta/2)} d[y]\, \delta(\langle \Psi_0|y\rangle) \times \left(\|\dot{x}_+\| - \langle \dot{\Psi}_0|y\rangle\right) e^{-\frac{1}{2}\langle y|H|y\rangle} \Big]^2, \qquad (16)$$

with the limits of integration over instanton positions chosen to avoid double counting. H is the operator in (15) with $\tau = 0$. The result (6) was obtained by neglecting $\langle \dot{\Psi}_0|y\rangle$ in (16) so that, in the same approximation, we infer from (6) the value of the term in brackets on the right-hand side of (16):

$$\|\dot{x}_+\| N \int_{y(\beta/2)=y(-\beta/2)} d[y]\, \delta(\langle \Psi_0|y\rangle)\, e^{-\frac{1}{2}\langle y|H|y\rangle} = -\frac{i}{2}\left(\frac{8}{\lambda\pi}\right)^{1/2}.$$

The factor $\frac{1}{2}$ occurs because x_+ and x_- give equal contributions to the result (6). It is fairly obvious that the calculation of the fluctuations about the remaining (quasi) saddle points x_{+-}, x_{-+} and x_{--} góes through exactly as above. Hence the total two-instanton contribution to Z, for small λ, is

$$Z_{++} + Z_{+-} + Z_{-+} + Z_{--} \simeq \frac{\beta^2}{2!}\left[-i\left(\frac{8}{\lambda\pi}\right)^{1/2} e^{-4/3\lambda}\right]^2.$$

Summing over all multi-instantons gives

$$Z(-\lambda) = \exp\left[-i\beta\left(\frac{8}{\lambda\pi}\right)^{1/2} e^{-4/3\lambda}\right]. \qquad (17)$$

The exponentiation of the one-instanton contribution to Z is expected in view of the previous calculations of Callan, Dashen and Gross[9] and Polyakov.[10] From (1), (2) and (17) we obtain

$$E_n \underset{n\to\infty}{\sim} (-1)^{n+1}\left(\frac{6}{\pi^3}\right)^{1/2}\left(\frac{3}{4}\right)^n \Gamma(n+1/2),$$

which, when multiplied by 4^n to correct for a different definition of g, is the result of Bender and Wu. Only quadratic fluctuations about multi-instantons and the saddle point at x(t) = 0 have been retained to arrive at (17). Non-Gaussian terms must be retained to calculate the non-leading contributions to E_n for large n.

In conclusion, it is reassuring that two different calculations within the general formalism of Lipatov yield identical results for the anharmonic oscillator and that these are in complete agreement with the calculation of Bender and Wu.

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