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Endomorphism rings of torsion-free modules over a complete discrete valuation ring

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§1. Introduction.

Our interest here will be to characterise the endomorphism rings of torsion-free reduced modules over complete discrete valuation rings, i.e. commutative principal ideal rings with exactly one prime element up to unit factors. A complete characterisation in ring-theoretic terms has been given by Liebert [8] but with a view to possible applications (see Goldsmith [5]) we approach the problem from the module-theoretic position making use of the concept of a basic submodule. We remark that in this approach, as in Liebert's, we require a certain topological property. In section 5 we show how to obtain a characterisation in ring-theoretic terms. This is, of course, equivalent to the characterisation in [8], but the conditions appear in somewhat different form.

We close this introduction by establishing some terminology. Throughout R will denote a complete discrete valuation ring with prime element p. If X is any torsion-free R-module then X may be topologized by taking the submodules p^nX as a basis of neighbourhoods of zero. This topology will be Hausdorff precisely if $\bigcap p^nX = 0$; since X is torsion-free this is equivalent to X being reduced. (For undefined terms we refer to Fuchs [3],[4] and Kaplansky [6].) The completion of X in this topology will be denoted by $\hat{X}$.

§2. Some properties of the endomorphism ring of a reduced torsion-free R-module.

We begin with a well-known result which shows the importance of a basic submodule of a reduced torsion-free R-module.

Lemma 2.1. If G is a reduced torsion-free R-module then G is a pure submodule of the completion $\hat{B}$ of any basic submodule B of G.

Proof:

By the proof of a theorem of Kaplansky ([6] Th. 22) we know that $\hat{G} = \hat{B}$ for any basic submodule B of G. Since G is reduced, G is pure in $\hat{G}$ and hence we have the result.

There is nothing canonical about the choice of the basic submodule B. We will just select and fix one basic submodule throughout and refer to this fixed basic submodule as B.

Lemma 2.2. If G is a pure submodule of $\hat{B}$ which contains B, then any endomorphism ϕ of G extends uniquely to an endomorphism $\hat{\phi}$ of $\hat{B}$.

Proof:

This is standard. See e.g. Fuchs [3] Th. 13.8.

In view of Lemma 2.2 we may, and do, regard endomorphisms of G as endomorphisms of $\hat{B}$. Let $E = E_R(\hat{B})$ and set $I(B) = \{\phi \in E | \hat{B}\phi \leq B\}$. It is clear that I(B) is a left ideal of E. I(B) will play a crucial role in our characterisation of endomorphism rings of reduced torsion-free R-modules. Recall that the finite topology on the endomorphism ring of an R-module is the topology introduced by Szele [13].

Lemma 2.3. If G is a pure submodule of $\hat{B}$ containing B then

(i) $I(B) \leq E(G)$

(ii) E(G) is a p-pure subalgebra of E

(iii) E(G) is complete in its finite topology.

Proof:

(i) is trivial while (ii) follows immediately from the fact that G is torsion-free and pure in $\hat{B}$. (iii) is well known, see e.g. Fuchs [4] Th. 107.1.

Remark: Since endomorphic images of complete R-modules are complete and B only contains complete modules of finite rank, we can see that every endomorphism in I(B) has finite rank. However, if $G \neq B$, there will be endomorphisms of G with finite rank which are not in I(B).

§3. A Galois Connection.

Let Σ be a p-pure subalgebra of E which contains I(B) and set $G(\Sigma) = \{b\phi | \phi \in \Sigma, b \in B\}$.

Lemma 3.1. If $x \in G(\Sigma)$ then there exists ψ in Σ and b in B such that $\hat{B} = \langle b \rangle \oplus K$ and $b\psi = x$, $K\psi = 0$.

Proof:

Choose any element b in B such that b has p-height zero. Then clearly $\langle b \rangle$ is a direct summand of $\hat{B}$ i.e. $B = \langle b \rangle \oplus K$, some K. Now if $x = a\phi$, $a \in B$, $\phi \in \Sigma$, let δ be the endomorphism of B sending b to a and annihilating K. Clearly $\delta \in I(B)$. Set $\psi = \delta\phi$, then ψ is in Σ since Σ is a subring and clearly ψ has the required properties.

Lemma 3.2. $G(\Sigma)$ is a p-pure submodule of $\hat{B}$ containing B.

Proof:

We show firstly that $G(\Sigma)$ is a submodule. Let z and w be in $G(\Sigma)$ and suppose $z = b_1 \phi_1$, $w = b_2 \phi_2$ where ϕ_i is in Σ and b_i is in B (i = 1, 2). Since a torsion-free R-module is fully transitive we can find an endomorphism δ with $b_1\delta = b_2$ or $b_2\delta = b_1$. Moreover δ can be chosen to be in I(B). Say $b_1\delta = b_2$. Then $w = b_1\delta\phi_2$ and since Σ is a subalgebra $\delta\phi_2 \in \Sigma$. Hence $z - w = b_1(\phi_1 - \delta\phi_2) \in G(\Sigma)$. If $r \in R$ then for any $x \in G(\Sigma)$ it is clear that $xr \in G(\Sigma)$. Thus $G(\Sigma)$ is a submodule and it is clear that it contains B.

Finally to show that $G(\Sigma)$ is p-pure, suppose $y \in G(\Sigma) \cap p^k\hat{B}$, say $y = p^k x$. Now applying Lemma 3.1 to y we get $y = b\psi = p^k x$ for some $b \in B$, $\psi \in \Sigma$. Now define a homomorphism ζ by setting $b\zeta = x$, $K\zeta = 0$. Then clearly $\psi = p^k \zeta \in \Sigma \cap p^k E = p^k \Sigma$. It follows by torsion-freeness that $\zeta \in \Sigma$, so that $x \in G(\Sigma)$. Thus $y \in p^k G(\Sigma)$ and this completes the proof.

§4. The First Characterisation.

Before proceeding to the characterisation we introduce a further topological concept.

Definition: An idempotent π in a ring Σ is said to be a finite idempotent if it is possible to write $\pi = \pi_1 + \dots + \pi_n$ where the π_i $(1 \le i \le n)$ are indecomposable mutually orthogonal idempotents in Σ.

We denote by $\Phi(\Sigma)$ the set of finite idempotents in the ring Σ. We now define a topology τ on Σ by taking as a basis of neighbourhoods of zero the sets $N_\pi = \{\eta \in \Sigma \mid \pi\eta = 0\}$ where $\pi \in \Phi(\Sigma)$. For an arbitrary ring there is no reason to suppose that this definition would yield a topology, however, in the case we are interested in viz. $I(B) \le \Sigma \le E$, we do in fact get a basis for a topology as is shown below.

Lemma 4.1. If $I(B) \le \Sigma \le E$ then the sets $N_\pi (\pi \in \Phi(\Sigma))$ form the basis of a neighbourhood system for a topology τ on Σ.

Proof:

If $\pi_1, \pi_2 \in \Phi(\Sigma)$ we must find $\pi \in \Phi(\Sigma)$ such that $N_\pi \le N_{\pi_1} \cap N_{\pi_2}$. However $\hat{B}\pi_1 + \hat{B}\pi_2$ is clearly a finite rank submodule of $G(\Sigma)$ and so if K is the pure submodule of $G(\Sigma)$ containing $\hat{B}\pi_1 + \hat{B}\pi_2$, then the projection π of $\hat{B}$ onto K satisfies $N_\pi \le N_{\pi_1} \cap N_{\pi_2}$. Moreover it follows from the fact that $I(B) \le \Sigma$, a subring, that $\pi \in \Phi(\Sigma)$. Indeed the topology is T_1 since

$$\bigcap_{\pi \in \Phi(\Sigma)} N_\pi = 0 \qquad \text{for if} \quad \phi \in \bigcap_{\pi \in \Phi(\Sigma)} N_\pi \quad \text{then}$$

$B\phi = 0$ and so by density of B in p-adic topology, $\phi = 0$ on $\hat{B}$ also.

Lemma 4.2. If G is a p-pure submodule of $\hat{B}$ containing B, then $E(G)$ is complete in the τ-topology which coincides with the Szele finite topology.

Proof:

The proof is similar to the result for p-groups, see e.g. Liebert [7] Proposition 3.2, or Fuchs [4] Th. 107.2.

In the light of Lemma 4.2 we may call the topology τ the finite topology.

Theorem 4.3. Let Σ be a p-pure subring of E containing $I(B)$, then if Σ is complete in the finite topology, $\Sigma = E_R(G(\Sigma))$.

Proof:

Since $G(\Sigma)$ is clearly a faithful right Σ-module we can identify $\Sigma \le E_R(G(\Sigma))$. Let $\zeta \in E_R(G(\Sigma))$ and let $\{F_i \mid i \in D\}$ be the set of finite subsets of $G(\Sigma)$ ordered so that $i \le j$ if and only if $F_i \le F_j$. The aim is to construct a net $\{\zeta_i\}_{i \in D}$ in Σ such that $F_i(\zeta - \zeta_i) = 0$. For then the net $\{\zeta_i\}$ is Cauchy in the finite topology and so has a limit $\eta \in \Sigma$. But then $B(\zeta - \eta) = 0$ and so $\zeta = \eta \in \Sigma$. Thus it will suffice to construct the net with the above properties.

Now given F_i we can embed it in a direct summand G_i of $\hat{B}$ where $G_i \le G(\Sigma)$ and has finite rank. Let π_i denote the projection of $\hat{B}$ onto G_i. Then we have

$$F_i(\zeta - \pi_i\zeta) = 0 \tag{1}$$

But $F_i \pi_i \le G_i = \langle g_1 \rangle \oplus \langle g_2 \rangle \oplus \dots \oplus \langle g_k \rangle$, say. Now $g_i\zeta \in G(\Sigma)$ and so

$g_i\zeta = b_i\phi_i$ some $b_i \in B$, $\phi_i \in \Sigma$. But since a reduced torsion-free R-module is fully transitive, there exist endomorphisms χ_i with $g_i\chi_i = b_i$ and χ_i annihilates the complement of $\langle g_i \rangle$. Clearly $\chi_i \in I(B)$ for each i. Let $\xi_i = \sum_{i=1}^{k} \chi_i\phi_i$; $\xi_i \in \Sigma$ and we have

$$F_i\pi_i(\xi_i - \zeta) = 0 \qquad (2)$$

Let $\zeta_i = \pi_i\xi_i$, then $\zeta_i \in \Sigma$ and from (1) and (2) we get

$$F_i(\zeta - \zeta_i) = 0.$$

Thus the net $\{\zeta_i\}$ has the required property and the result follows.

Combining Lemma 2.3 and the above result we obtain our first characterisation.

Theorem 4.4. A ring Σ is the endomorphism ring of a reduced torsion-free R-module if and only if there exists a free R-module B such that Σ is isomorphic to a p-pure subring of $E_R(\hat{B})$ containing I(B) and Σ is complete in its finite topology.

§5. A ring theoretic characterisation.

From the viewpoint of constructing examples the condition in Th. 4.3 that $I(B) \le \Sigma \le E(\hat{B})$ is reasonably satisfactory but from the viewpoint of abstract ring theory the left ideal I(B) is totally unnatural. We can however give ring theoretic conditions which ensure that we are working only with subrings of the endomorphism ring of a complete module $\hat{B}$ which contain I(B). For this section we shall suppose that E is a unital associative ring.

Lemma 5.1. If E is a ring with idempotents e and f and $eE \hat{=} fE$ as E-modules, then eEf is a free eEe-module in one generator. Moreover eEfE = eE.

Proof:

See Liebert [9], Lemma 2.3.

Lemma 5.2. Let E be a ring such that

(i) If e is a minimal idempotent of E then eEe is a complete discrete valuation ring.

(ii) If e and f are any two minimal idempotents then $eE \hat{=} fE$ as E-modules.

(iii) E contains a set of minimal orthogonal idempotents $\{e_i | i \in I\}$ and a set of nilpotent elements $\{\varepsilon_{ij} | i,j \in I\}$ such that

(a) $e_i\varepsilon_{jk} = \delta_{ik}e_k$, $\varepsilon_{ij}\varepsilon_{k\ell} = \delta_{jk}\varepsilon_{ie}$ (δ_{ij} a Kronecker delta)

(b) $e_iE \le (\oplus E e_i)^\wedge$, the completion being with respect to the p-adic topology, where p is the generator of the Jacobson radical of eEe.

(iv) $\bigcap_{i\in I} \mathrm{ann}_r(e_i) = 0$ where $\mathrm{ann}_r(e_i) = \{f \in E | e_if = 0\}$.

Then there exists a free module F over a complete discrete valuation ring R such that $I(F) \le E \le E_R(\hat{F})$.

Proof:

Choose a minimal idempotent e and set $F = \bigoplus_{i\in I} eEe_i$. Clearly F is a free R-module since by Lemma 5.1. eEe_i is a free R-module of rank 1. Let $\hat{F}$ denote the p-adic completion of F. Condition (iv) ensures that $\hat{F}$ is a faithful E-module so we may regard E as a subring of $E_R(\hat{F})$.

Now if $\phi \in I(F)$ then ϕ has finite rank and so $\phi = e_1\phi + \dots + e_k\phi$ some k. Moreover for each i, there is an n(i) in I such that $e_i\phi$ has image contained in $\bigoplus_{i=1}^{n(i)} eEe_j$. Choosing a maximum such n(i) for i = 1, ... k we see that

$$(eEe_i)\phi \in \bigoplus_{i=1}^{N} eEe_j \quad \text{and so}$$

$$e_i\phi = \sum \varepsilon_{ij} \quad \text{where this summation is finite.}$$

So by (iii) $e_i\phi \in E$ and hence $\phi \in E$. Thus $I(F) \le E$ as required.

Theorem 5.3. An abstract ring E is isomorphic to the endomorphism ring of a torsion-free module over a complete discrete valuation ring if and only if

(i) If e is a minimal idempotent of E then eEe is a complete discrete valuation ring.

(ii) If e and f are any two minimal idempotents then $eE \cong fE$ as E-modules.

(iii) E contains a set of minimal orthogonal idempotents $\{e_i | i \in I\}$ and a set of nilpotent elements $\{\varepsilon_{ij} | i,j \in I\}$ such that

(a) $e_i\, \varepsilon_{jk} = \delta_{ik}\, e_k$, $\varepsilon_{ij}\, \varepsilon_{k\ell} = \delta_{jk}\, \varepsilon_{i\ell}$

(b) $e_iE \le (\oplus Ee_i)^{\wedge}$, the completion being with respect to the p-adic topology where p is the generator of the Jacobson radical of eEe.

(iv) E is complete in its finite topology.

(v) If I denotes the ideal of finite endomorphisms of $\oplus\, eEe_i$ then if $\zeta \in E$ and $\zeta I \le p^kI$ then $\zeta \in p^kE$.

Proof:

Since completeness of the finite topology includes the Hausdorff condition then (iv) above contains part (iv) of Lemma 5.2. In view of Th. 4.4. it will be sufficient for the "if" part of the proof to show that (v) guarantees the p-purity of E.

The necessity is clear except that we must show that p-purity of a subring E of $E_R(\hat{F})$ implies condition (v). The proof is therefore completed by the following result.

Lemma 5.4. If F is a free R-module and $I(F) \le E \le E_R(\hat{F})$ then E is p-pure in $E(\hat{F})$ if and only if for any ζ in E, $\zeta I(F) \le p^kI(F)$ implies $\zeta \in p^kE$.

Proof:

Let $\zeta \in p^kE(\hat{F}) \cap E$, then $\zeta I(F) = p^k\chi I(F)$ for some $\chi \in E(\hat{F})$. Since I(F) is a left ideal of $E(\hat{F})$ we have $\zeta I(F) \le p^kI(F)$ and so by assumption $\zeta \in p^kE$. Thus $p^kE(\hat{F}) \cap E = p^kE$.

Conversely suppose E is p-pure and $\zeta I(F) \le p^kI(F)$. Let π_i denote the projection of $\hat{F}$ onto the i^{th} summand of F. Then $(x\zeta)\pi_i = p^kx\chi$ for some $\chi \in I(F)$ since $\pi_i \in I(F)$. But this is true for any $x \in \hat{F}$ and each i, so $\hat{F}\zeta \le p^k\hat{F}$. If we define $\psi : \hat{F} \to \hat{F}$ by $x\psi = p^{-k}(x\zeta)$ then the torsion-freeness of $\hat{F}$ ensures that ψ is well defined. Clearly $p^k\psi = \zeta$. Thus $\zeta = p^k\psi \in E \cap p^kE(\hat{F}) = p^kE$. So $\zeta \in p^kE$.

Remark: The techniques used so far do not depend on the commutivity of the complete discrete valuation ring and all arguments can be easily generalised to include non-commutative complete discrete valuation rings.

§6. Further topological considerations.

Since the characterisation we have obtained requires the use of one topological property we try to exploit this to the full. The situation is remarkably similar to the corresponding problem for endomorphism rings of p-groups (see Pierce [10]) but is simpler in that we require no topology other than the finite topology of annihilators already introduced.

Let μ be an infinite ordinal and set $B = \bigoplus_{i<\mu} R\, e_i$. Let π_i denote the projection $B \to R\, e_i$.

Lemma 6.1. If $\zeta \in I(B)$ then $\lim_{k\to\mu} \sum_{s\le k} \pi_s\zeta = \zeta$ where the limit is taken with respect to the finite topology of I(B).

<u>Proof</u>:

Let q be a finite idempotent in I(B) then there is $k < \mu$ such that $q\,\delta_k = q$ where $\delta_k = \sum_{s \le k} \pi_s$.

Now
$$\zeta - \sum_{s \le k} \pi_s \zeta = (1 - \delta_k)\zeta .$$
Thus
$$q(\zeta - \sum_{s \le k} \pi_s \zeta) = q(1 - \delta_k)\zeta = q\,\delta_k(1 - \delta_k)\zeta = 0.$$

<u>Theorem 6.2.</u> Let Σ be a ring which contains I(B) as a faithful left ideal. Suppose that the mapping $\phi \to \zeta\phi$, where $\phi \in I(B)$, $\zeta \in \Sigma$, is continuous in the finite topology of I(B) for all $\zeta \in \Sigma$. Then there is a ring isomorphism of Σ into $E(\hat{B})$ which is the identity on I(B).

<u>Proof</u>:

For $\zeta \in \Sigma$ and $x \in \hat{B}$ define
$$x\lambda_\zeta = \sum_{i < \mu} x(\zeta\pi_i)$$

Since $ht(x(\zeta\pi_i)) \ge ht(x\pi_i)$ and $x \in \hat{B}$ has at most countably many non-zero components, it is clear that the term on the r.h.s. represents an element of $\hat{B}$. Clearly λ_ζ is an endomorphism of $\hat{B}$. Moreover if $\zeta \in I(B)$ then
$$x\lambda_\zeta = \sum_{i<\mu} x(\zeta\pi_i) = \sum_{i<\mu} (x\zeta)\pi_i = x\zeta .$$

Thus the mapping $\lambda : \Sigma \to E(\hat{B})$ acts as the identity on I(B).

By the distribution law in Σ we have that $\lambda_{\zeta-\eta} = \lambda_\zeta - \lambda_\eta$ for any $\zeta,\eta \in \Sigma$. To show that λ is a ring homomorphism it remains to show that $\lambda_{\zeta\eta} = \lambda_\zeta\lambda_\eta$.

Now by definition we have $x\lambda_{\zeta\eta} = \sum_{i<\mu} x(\zeta\eta\pi_i)$.

But by Lemma 6.1. $\lim_{k \to \mu} \sum_{s \le k} \pi_s \eta\pi_i = \eta\pi_i$ and since, by assumption, multiplication is continuous we have that
$$\lim_{k \to \mu} \sum_{s \le k} \zeta\pi_s\eta\pi_i = \zeta\eta\pi_i$$

So
$$x\lambda_{\zeta\eta} = \sum_{i < \mu} x(\zeta\eta\pi_i) = \sum_{i < \mu} x(\lim_{k \to \mu} \sum_{s \le k} \zeta\pi_s\,\eta\pi_i)$$

But if $x \in B$ then x has only finitely many non-zero components and so we have
$$x\lambda_{\zeta\eta} = \sum_{i < \mu} (\sum_{k < \mu} x(\zeta\pi_k)(\eta\pi_i))$$
$$= x\lambda_\zeta\cdot\lambda_\eta .$$

So $\lambda_{\zeta\eta}$ agrees with $\lambda_\zeta\lambda_\eta$ on B and so by density of B in $\hat{B}$ (in p-adic topology) we have $\lambda_{\zeta\eta} = \lambda_\zeta\,\lambda_\eta$. Thus λ is a ring homomorphism. Finally suppose $\lambda_\zeta = 0$ for some $\zeta \in \Sigma$. Now if $\eta \in I(B)$, we have $\zeta\eta \in I(B)$ and so $\zeta\eta = \lambda_{\zeta\eta} = \lambda_\zeta\,\lambda_\eta = 0$. Since I(B) is a faithful ideal we must have $\zeta = 0$. Thus λ is a ring isomorphism. This completes the proof.

We may combine the results of §4 to obtain the following characterisation.

<u>Theorem 6.3.</u> Let B be a free R-module of infinite rank, where R is a complete discrete valuation ring. Let Σ be a ring which contains I(B) as a faithful left ideal. Assume the following conditions hold:-

(a) Left multiplication by elements of Σ is a continuous homomorphism of I(B) in the finite topology on I(B).

(b) If $\zeta \in \Sigma$ and $I(B) \le p^k I(B)$ then $\zeta \in p^k\Sigma$.

(c) Σ is complete in its finite topology.

Then there is an R-module G with $B \le G \le \hat{B}$, G p-pure in $\hat{B}$ and an isomorphism λ of Σ onto E(G) such that λ is the identity on I(B).

Remark: It is clear that $I(B)$ is in fact the ideal of finite endomorphisms of the free R-module B and so we could extend Theorem 6.3. to give a more ring theoretic flavour to the result. This could be achieved by using techniques similar to Liebert [9]. We leave the formulation of this result to the reader.

§7. An Example.

The similarity between the characterisation of endomorphism rings of torsion-free modules over a complete discrete valuation ring and endomorphism rings of p-groups is clear (cf. Pierce [10]). Now in the theory of p-groups Pierce, Beaumont and Corner (see [11], [1] and [2]) have exhibited groups with endomorphism rings which are ring split extensions of a prescribed ideal by a complete discrete valuation ring. We use Theorem 4.3. to yield a similar result here.

Example: We remark that the ring split extension $R \oplus I(B)$ cannot be the endomorphism ring of an R-module containing a free R-module B. However, by analogy with $I(B)$, we define, for each pure submodule G of $\hat{B}$ which contains B, an ideal $I(G)$ in the endomorphism ring of G by setting $I(G) = \{\phi \in E(G) | \hat{B}\hat{\phi} \leq G\}$. (Here $\hat{\phi}$ denotes the unique extension of ϕ to $\hat{B}$.)

In the particular case when G is a maximal pure submodule of $\hat{B}$ containing B we show that $R \oplus I(G)$ is the endomorphism ring of some R-module. (Indeed it is the endomorphism ring of G itself.) Trivially $I(B) \leq R \oplus I(G) \leq E(\hat{B})$, while if $r + \theta = p^k\chi$ some $\chi \in E(\hat{B})$, $\theta \in I(G)$ and $r \in R$, then $p^k | r$ (otherwise θ would be a unit). It follows easily that $\theta \in p^k I(G)$ and so $R \oplus I(G)$ is p-pure in $E(\hat{B})$.

Finally if $\{\zeta_i\}$ is a Cauchy net in $R \oplus I$, we show that it has a limit in $R \oplus I$. Let $\{\pi_j\} i \in I$ denote the projections of $\hat{B}$ onto the rank one basis elements of B. Then $\pi_i(\zeta_k - \zeta_j) = 0$ for all $j, k \geq n(\pi_i)$. Let $\pi_i \zeta^i$ denote the common

value of $\pi_i \zeta_k$ for $k \geq n(\pi_i)$. Then $\zeta = \sum_{i \in I} \pi_i \zeta^i$ is a well defined element of $E(\hat{B})$. It is clearly the limit of the Cauchy net and so it remains to show that $\zeta \in R \oplus I(G)$. But for any $x \in \hat{B}$, there is a ζ_x in the Cauchy net such that $x(\zeta - \zeta_x) = 0$. (This follows from density of G in $\hat{B}$ in the p-adic topology.) So for each $x \in \hat{B}$ we have $x\zeta = x\zeta_x = x(r_x + \theta_x)$ some $r_x \in R$, $\theta_x \in I(G)$. However the maximality of G and its invariance under ζ imply that we need only choose one such r_x, call it r, so that $y(\zeta - r) \in G$ for all $y \in \hat{B}$. Thus $\zeta \in R \oplus I(G)$.

Remark: The situation here is easier than the problem for p-groups since the ideal occurring in the splitting here is not as tightly prescribed as the ideal of small endomorphisms for p-groups. This result has been modified by the author in [5] to deal with the case where B is countable but no results analagous to Shelah's results [12] on p-groups have yet been found.

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