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Smooth Perturbations of One-Parameter Groups on W*-Algebras

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Abstract.

We are concerned with the smooth perturbation of one-parameter groups on von Neumann algebras.

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1. *Introduction*

We investigate the perturbation, similarity and scattering of weakly continuous one-parameter groups of ultraweakly continuous operators on W*-algebras.

In a previous paper [2] we showed how to extend Lin's [5] time dependant reflexive Banach space theory of the smooth perturbation of generators of strongly continuous semi-groups to the possibly non-reflexive situation. We then saw that this theory could handle a particular smoothness condition (namely $p = 1$) on the preduals of W*-algebras. In this paper we are concerned with another class of smooth perturbations (namely $1 < p < \infty$). However, due to technical problems, we apply our method only to type I - W*-algebras with atomic centre.

If σ_t is a weakly continuous one parameter group of ultraweakly continuous linear maps on a von Neumann algebra M, let iT denote the infinitesimal generator of the strongly continuous one-parameter group $(\sigma_t)_*$ acting on the predual M_*. For various unbounded operators β and α, acting on M, we seek for small complex K, a family of weakly continuous one parameter groups σ_t^k, such that the infinitesimal generator iT(K) of $(\sigma_t^k)_*$ can be interpreted as the formal symbol $i(T + k\alpha_*\beta_*)$. As in [1,2] our method is based on the theory of Lin for the problem at the hilbert space level [5], using the one parameter groups σ_t themselves, rather than the resolvents $R(\lambda, T)$ as Kato does [4].

β and α can represent left multiplication operators L_B, L_A respectively, where B and A are unbounded operators affiliated with M. In general, if σ_t is a group of *-automorphisms, the perturbation σ_t^k will not be a group of *-automorphisms for small (real) K. By considering another perturbation of multiplication on the right, (R_{B*}, R_{A*}), it is sometimes possible to combine these two perturbations to give one, namely τ_t^k, which is a weakly continuous group of *-automorphisms of M for small real K. Moreover, the infinitesimal generator of $(\tau_t^k)_*$ represents the formal symbol $i[T + k((\alpha)_*\beta_* - (\bar{\alpha})_*(\bar{\beta})_*)]$.

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2. *Notation.*

If X and Y are Banach spaces, $\mathcal{C}(X, Y)$ (respectively $B(X, Y)$, $C(X, Y)$) will denote the closed, densely defined (respectively bounded, compact) linear operators from X into Y. We put $\mathcal{C}(X) = \mathcal{C}(X, X)$ etc.; and $\mathcal{C}_+(X)$ to be those T in $\mathcal{C}(X)$ such that $-iT$ generates a strongly continuous semigroup. Then let $\mathcal{C}_-(X) = -\mathcal{C}_+(X)$. If $D \in \mathcal{C}(X,Y)$ then D^* is a linear map from Y^* into X^* which is closed and densely defined for the weak *-topologies [3, 7]. Similarly, let C be a linear map from Y^* into X^* which is closed and densely defined for the weak *-topologies. If $x \in X$, we say $x \in D(C_*)$ if there exists y in Y such that

$$\langle C\rho, x \rangle = \langle \rho, y \rangle \qquad \rho \in D(C).$$

and define $C_* x = y$. Then $C_* \in \mathcal{C}(X,Y)$. If T is any map with domain D in X and codomain in Y, then the extended norm is defined by setting

$$\| Tx \| = \infty \qquad \text{if} \quad x \in X \backslash D.$$

3. *Preliminaries.*

In this section we develop some new notation and tools for our perturbation theory, which we shall find particularly useful to handle the smoothness conditions we impose on our system.

For simplicity and clarity of notation, we will only consider groups, rather than semigroups as in [2, 5].

Now suppose $\mathcal{H}$ is a separable hilbert space and $\pm T \in \mathcal{C}_+(\mathcal{H})$. If $A \in \mathcal{C}(\mathcal{H})$ is $(T, -T, q)$ [2, 5] smooth where $1 < q < \infty$, we have seen in [2] the difficulties which arise when considering smoothness in this system at an algebra level. However, if we let

$$\sigma_t(x) = e^{-itT} x e^{itT}, \quad x \in B(\mathcal{H}),$$

then for all x in $B(\mathcal{H})$, $\xi \in \mathcal{H}$, $\sigma_t(x) e^{-itT}\xi \in D_A$ a.e. in $\mathbb{R}^+$,

and $$\left\{ \int_0^\infty \| A \sigma_t(x) e^{-itT}\xi \|^q dt \right\}^{1/q} \leq |A| \, \|x\| \, \|\xi\| .$$

Our perturbation theorem is based on these "weak smoothness" conditions, and a few calculations by the reader will show that we have raised [5, Theorem 3.1] to the algebra level. It is useful to keep this motivation in mind.

We are thus led to study spaces such as

$$\mathcal{Y} = B(\mathcal{H}, L^q(\mathbb{R}^+; \mathcal{H})) \qquad \text{where } 1 < q < \infty .$$

Then $\mathcal{Y}$ is a dual Banach space, with predual

$$\mathcal{X} = \overline{\mathcal{H} \otimes^\vee L^p(\mathbb{R}^+; \mathcal{H})},$$ the projective tensor product, where $1/p + 1/q = 1$.

Again $\mathcal{X}$ is a dual Banach space if we take

$$\mathcal{X}_* = \overline{\mathcal{H} \otimes^\wedge L^q(\mathbb{R}^+; \mathcal{H})},$$ the injective tensor product.

Let V be the space of all operators on $\mathcal{H}$. An element of V is not necessarily closed, densely defined or even linear. Then $\mathcal{Y}$ can be identified with the space of V-valued functions f on $\mathbb{R}^+$ satisfying

(3.1) If $\xi \in \mathcal{H}$, then $\xi \in D(f(s))$ a.e. Note this null set may depend on ξ.

(3.2) If $\alpha, \beta \in \mathbb{C}$, and $\xi, \eta \in \mathcal{H}$, then

$$f(s)(\alpha\xi + \beta\eta) = \alpha f(s)\xi + \beta f(s)\eta \quad \text{a.e. (depending on } \alpha, \beta, \xi, \eta).$$

(3.3) For each $\xi \in \mathcal{H}$, $s \mapsto f(s)\xi$ is measurable, and there exists a finite M independent of ξ satisfying

$$\int_0^\infty \| f(s)\xi\|^q \, ds \leq M^q \|\xi\|^q \quad , \quad \xi \in \mathcal{H} \tag{3.4}$$

If $f \in \mathcal{Y}$, $\|f\|$ is the infimum of all M satisfying (3.4).

Now let M be a von Neumann algebra acting on the separable hilbert space $\mathcal{H}$. Then M is ultraweakly countably generated in the sense that there is a countable subset R of M, such that the linear span of R is $\sigma(M,M_*)$ dense in M [6]. Define $\mathcal{Y}_M$ to be the closed subspace of $\mathcal{Y}$ consisting of these elements which commute with the action of M' on $\mathcal{H}$ and $L^q(\mathbb{R}^+; \mathcal{H})$. i.e. $f \in \mathcal{Y}$ is in $\mathcal{Y}_M$ iff for each ξ in $\mathcal{H}$, and m' in M', that

$$f(s)(m'\xi) = m' f(s)\xi \quad \text{a.e. (depending on } m' \text{ and } \xi).$$

Now $\mathcal{Y}_m$ is a $\sigma(\mathcal{Y},\mathcal{X})$ closed subspace of $\mathcal{Y}$. Since if $f_\alpha \to f \ \sigma(\mathcal{Y},\mathcal{Z})$, where $f_\alpha \in \mathcal{Y}_M$, $f \in \mathcal{Y}$; then for all $\xi, \eta \in \mathcal{H}$, $h \in L^p(\mathbb{R}^+)$, $m' \in M'$,

$$\begin{aligned}\int \langle f(s)m'\xi, \eta\rangle h(s)\, ds &= \lim_\alpha \int \langle f_\alpha(s)m'\xi, \eta\rangle h(s)\, ds \\ &= \lim_\alpha \int \langle m'f(s)\xi, \eta\rangle h(s)\, ds \\ &= \int \langle m'f(s)\xi, \eta\rangle h(s)\, ds.\end{aligned}$$

Thus $f \in \mathcal{Y}_M$.

Hence $(\mathcal{Z}/\mathcal{Y}_M^\circ)$ is isometrically isomorphic to $\mathcal{Y}_M$, where $\mathcal{Y}_M^\circ$ denotes the polar of $\mathcal{Y}_M$ in $\mathcal{Z}$. Moreover, if i_M denotes the canonical injection of $\mathcal{Y}_M$ in $\mathcal{Y}$, and if $\mathcal{Z}_M$ is defined as $i_M^* \mathcal{Z}$ (the restriction of $\mathcal{Z}$ to $\mathcal{Y}_M$), then $\mathcal{Z}_M$ is isometrically isomorphic to $\mathcal{Z}/\mathcal{Y}_M^\circ$. Thus $\mathcal{Z}_M$ is a closed subspace of $\mathcal{Y}_M^*$, and if $z_o \in \mathcal{Z}_M$, then

(3.5) $\quad \| z_o \| = \inf\{ \| x \| : x \in \mathcal{Z} \text{ s.t. } i_M^*(x) = z_o \}$.

Now if $z = \sum_{i=1}^{\infty} \xi_i \otimes \bar{h}_i$ is in $\mathcal{Z}$, where $\xi_i \in \mathcal{H}$, $h_i \in L^p(\mathbb{R}^+; \mathcal{H})$, for $i = 1, 2, \ldots\ldots$ and $\sum_i \| \xi_i \| . \| h_i \| < \infty$, then

$$\left\{ \int \left(\sum \|\xi_i\| \, \|h_i(s)\| \right)^p ds \right\}^{1/p} \leq \sum_{i=1}^{\infty} \left\{ \int \left(\|\xi_i\| \, \|h_i(s)\| \right)^p ds \right\}^{1/p}$$

$$= \sum \|\xi_i\| \, \|h_i\| < \infty .$$

Hence $\sum \| \xi_i \| \; \| h_i(s) \| < \infty$ a.e. in $\mathbb{R}^+$, and we can define

$$z(s) = \sum \xi_i \otimes h_i(s)^-$$

a.e. in $\mathcal{H} \otimes^{\vee} \bar{\mathcal{H}} = T(\mathcal{H})$, the trace class operators on $\mathcal{H}$.

Note that z(s) is uniquely determined by z a.e., $s \mapsto z(s)$ is measurable, and

(3.6) $\quad \{ \int \| z(s) \|^p ds \}^{1/p} \leq \| z \|$.

If $f \in \mathcal{Y}$ is bounded a.e. (e.g. if f is actually in $L^q(\mathbb{R}^+; B(\mathcal{H}))$ then

$$\langle f, z \rangle = \sum_i \int \langle f(s)\xi_i, h_i(s) \rangle \, ds = \int \langle f(s), z(s) \rangle \, ds.$$

Similarly, we consider $\mathcal{Z}_M$. If $x \in \mathcal{Z}_M$, then $x = i_M^*(z)$, for some z in $\mathcal{Z}$. Then $z(s) \in T(\mathcal{H})$ determines an element x(s) in M_* a.e. Moreover x(s) is uniquely determined by x a.e., and $x(\cdot)$ is an element of $L^p(\mathbb{R}^+; M_*)$. It is clear that whenever $x = i_M^*(z_1)$, where $z_1 \in \mathcal{Z}$, that

$$\langle x(s), m \rangle = \langle z_1(s), m \rangle \quad \text{a.e. in } \mathbb{R}^+, \text{ for any } m \text{ in } M.$$

REMARK 3.7. We thus have a linear map $x \mapsto x(\cdot)$ from $\mathfrak{Z}_M$ into $L^P(\mathbb{R}^+; M_*)$. This is norm reducing by 3.5 and 3.6:

$$\| x(\cdot) \| \leq \| x \| \tag{3.8}$$

Then if $f \in \mathcal{Y}_M$, and $f(s) \in M$ a.e. (e.g. if f is actually in $L^q(\mathbb{R}^+; M)$), then for all z in $\mathfrak{Z}_M$

$$\langle f, z \rangle = \int \langle f(s), z(s) \rangle \, ds.$$

If $f \in \mathcal{Y}_M$, $z \in \mathfrak{Z}_M$, the notation $f(s) \in V$, $z(s) \in M_*$ will be used extensively throughout this paper.

We shall see during the proof of Theorem 5.1. that technical problems lead us to restrict our attention to consider only von Neumann algebras which satisfy the following hypothesis.

HYPOTHESIS 3.9.

We say M satisfies this hypothesis if whenever $\phi(s) \in M_*$ a.e. in $\mathbb{R}^+$, and x_n, n = 1, 2, ... is a sequence in $\mathfrak{Z}_M$ such that

i) $x_n(s) = \phi(s)$ a.e. in $[o, n]$, n = 1, 2, ...

ii) $\| x_n \| \leq T < \infty$, for all n,

then there exists an x in $\mathfrak{Z}_M$ satisfying

a) $x(s) = \phi(s)$ a.e. in $\mathbb{R}^+$.

b) $\| x \| \leq T$.

PROPOSITION 3.10.

Suppose $\mathcal{H} = \bigoplus_{i=1}^{\infty} \mathcal{H}_i$ *, where* $\mathcal{H}_i$ *are separable hilbert spaces; then* $M = \bigoplus_i B(\mathcal{H}_i)$ *satisfies hypothesis* 3.9.

PROOF.

It is clear that $\mathcal{Y}_M = \bigoplus_{i=1}^{\infty} \mathcal{Y}_{B(\mathcal{H}_i)}$ where $\|(f_i)\| = \sup\|f_i\|$, $f_i \in \mathcal{Y}_{B(\mathcal{H}_i)}$,

and $\mathcal{Z}_M = \bigoplus_{i=1}^{\infty} \mathcal{Z}_{B(\mathcal{H}_i)}$ where $\|(z_i)\| = \sum_i \|z_i\|$, $z_i \in \mathcal{Z}_{B(\mathcal{H}_i)}$.

Now $\mathcal{Z}_{B(\mathcal{H}_i)} = (X_i)^*$, where $X_i = \mathcal{H}_i \otimes^{\wedge} \overline{L^q(\mathbb{R}^+; \mathcal{H}_i)}$, $i = 1, 2, \ldots$.

Suppose $\phi(s) \in M_*$ a.e., and $x_n = (x_n^i)_{i=1}^{\infty}$ is a sequence in $\mathcal{Z}_M$ such that

i) $x_n(s) = \phi(s)$ a.e. in $[o,n]$

ii) $\|x_n\| \leq T < \infty$ all n.

Then $x_n^m \in (X_m)^*$ m, n = 1, 2, ...

Hence there exists a subsequence n_k and x^m in $(X_m)^*$ such that

$$x_{n_k}^m \longrightarrow x^m \quad \sigma(X_m^*, X_m) \quad \text{as } k \to \infty .$$

Thus for each finite N

$$(x_{n_k}^m)_{m=1}^N \longrightarrow (x^m)_{m=1}^N \quad \sigma\left(\left(\bigoplus_{m=1}^N X_m\right)^*, \bigoplus_{m=1}^N X_m\right) \quad \text{as } k \to \infty .$$

Thus $\sum_{m=1}^N \|x^m\| \leq T$ for all N, and hence

$$x = (x^m) \in \mathcal{Z}_M = \bigoplus_{m=1}^{\infty} (X_m)^* , \quad \text{with } \|x\| \leq T .$$

Moreover $\langle x^m, \psi \rangle = \lim_{k \to \infty} \langle x_{n_k}^m, \psi \rangle$

for all $\psi \in X_m = \mathcal{H}_m \otimes^{\wedge} \overline{L^q(\mathbb{R}^+; \mathcal{H}_m)}$.

Using property i) above, it follows easily that

$$\phi(s) = (x^m(s)) \quad \text{a.e. in } \mathbb{R}^+ ,$$

i.e. $\phi(s) = x(s)$ a.e.

4. *Smoothness in operator algebras.*

Having set up our apparatus, we proceed to make our smoothness definitions. We will use the notation of the previous section with q fixed in $(1,\infty)$. Let M be a von Neumann algebra acting on a separable hilbert space $\mathcal{H}$, (and thus is ultraweakly countably generated). Let σ_t be an ultraweakly continuous one-parameter group of ultraweakly continuous bounded linear maps on M (i.e. each σ_t is a $\sigma(M, M_*)$ continuous operator on M, and $t \mapsto \langle \sigma_t(x), \phi \rangle$ is continuous for all x in M, ϕ in M_*). For brevity, we will always refer to such a group as a weakly continuous group on M. We define iT to be the infinitesimal generator of $(\sigma_t)_*$, which is a strongly continuous group on the predual M_* . We will eventually perturb such a one-parameter group. This perturbation of T will be of the form $k\alpha_*\beta_*$ where k is complex and small, and β, α are smooth operators on M to be defined. (For simplicity, we take the domain of β and the codomain of α to be in M.)

In order to have smooth conditions which are practical, we introduce an auxiliary weakly continuous one parameter group Γ_t on M, which satisfies the following:

(4.1) There exists a family $\{u_t : t \in \mathbb{R}\}$ in $B(\mathcal{H})$, such that u_t is invertible a.e. and

i) $\Gamma_t(x) = u_t x\, u_t^{-1}$ a.e. , $\forall\ x \in B(\mathcal{H})$.

ii) $\| u_t \|$, $\| u_t^{-1} \| \le M_0 < \infty$, for all t in $\mathbb{R}$.

We then let $v_t = u_t^{-1}$.

Let $A \in \mathcal{C}(\mathcal{H})$ be affiliated with M.

DEFINITION 4.2.

We define the left multiplication operator $\alpha_A = \alpha$ on M, associated with A, as follows:

We put $D_\alpha = \{x \in M : x\mathcal{H} \subseteq D_A\}$, and $\alpha x = Ax$ for $x \in D_\alpha$.
Then $\alpha : D_\alpha \to M$ is a linear operator on M which is closed and densely defined for the ultraweak topology.

DEFINITION 4.3.

We say α is $(\sigma, \Gamma, q, +)$ smooth if

1) There exists $Y \in B(M, \mathcal{Y}_M)$

2) For all u in M, ξ in $\mathcal{H}$, $\sigma_{-t}(u)v_t\xi \in D_A$ a.e. in $\mathbb{R}^+$ (depending on u, ξ) such that

$$[Y(u)]\,(s)\,\xi = u_t A\, \sigma_{-t}(u) v_t \xi \quad \text{a.e. in } \mathbb{R}^+.$$

We then define $|\alpha|_{(\sigma,\,\Gamma,\,q,\,+)}$ to be $\|Y\|$.
$Y(u)$ is formally written as $\Gamma_t(A\sigma_{-t}(u))$.

Similarly, we define $(\sigma, \Gamma, q, -)$ smoothness with associated function $\hat{Y}$ in $B(M, \mathcal{Y}_M)$. If α is $(\sigma, \Gamma, q, \pm)$ smooth, we say α is (σ, Γ, q) smooth and set

$$|\alpha|_{(\sigma,\Gamma,q)} = \max\{\, |\alpha|_{(\sigma,\Gamma,q,+)}\,,\ |\alpha|_{(\sigma,\Gamma,q,-)} \,\}.$$

Let $\beta : M \to M$ be a linear operator on M, which is closed and densely defined for the ultraweak topology. p is given by $1/p + 1/q = 1$.

DEFINITION 4.4.

We say β_* is $(\sigma, \Gamma, p, +)_*$ smooth if for each $r > 0$

i) There exists $Z_r \in B(M_*, \mathcal{Z}_M)$ with $\sup_{r>0} \|Z_r\| < \infty$

ii) For each $\phi \in M_*$, $e^{-itT}\phi \in D_{\beta_*}$ a.e. in $\mathbb{R}^+$

such that

$$[Z_r(\phi)]\,(s) = (\Gamma_{-s})_*\, \beta_*\, e^{-i(r-s)T}\phi\ \chi_{(o,r)}(s).$$

We then define $|\beta_*|_{(\sigma,\Gamma,p,+)_*} = \sup_{r>o} \| Z_r \|$

Similarly. we define $(\sigma, \Gamma, p, -)_*$ smoothness, with associated functions $\hat{Z}_r$ in $B(M_*, \mathcal{Z}_M)$. If β_* is $(\sigma, \Gamma, p, \pm)_*$ smooth we say β_* is $(\sigma, \Gamma, p)_*$ smooth and set

$$|\beta_*|_{(\sigma,\Gamma,p)_*} = \max \{ |\beta_*|_{(\sigma,\Gamma,p,+)_*} , |\beta_*|_{(\sigma,\Gamma,p,-)_*} \} .$$

We take $Z_o = \hat{Z}_o = 0.$

Now fix f in $\mathcal{Y}_M$, $r > o$ and suppose β_* is $(\sigma, \Gamma, p)_*$ smooth. Then for all ϕ in M_*,

$$| \langle f, Z_r(\phi) \rangle | \leq \| f \| \; |\beta_*| \; \| \phi \| .$$

Moreover $\phi \to \langle f, Z_r(\phi) \rangle$ is linear. Hence there exists $m^f(r)$ in $M = (M_*)^*$ such that

(4.5) $\langle m^f(r), \phi \rangle = \langle f, Z_r(\phi) \rangle , \quad \phi \in M_*, \; r \geq o, \; f \in \mathcal{Y}_M ;$

and similarly $\hat{m}^f(r)$ in M satisfying

(4.6) $\langle \hat{m}^f(r), \phi \rangle = \langle f, \hat{Z}_r(\phi) \rangle , \quad \phi \in M_*, \; r \geq o, \; f \in \mathcal{Y}_M .$

DEFINITION 4.7.

If $A \in \mathcal{C}(\mathcal{H})$ is affiliated with M, $\alpha = \alpha_A$, and β_* is $(\sigma, \Gamma, p)_*$ smooth, we define an operator symbolically denoted by $(\alpha\sigma_{-(\cdot)}\beta)\theta$ as follows:

If $f \in \mathcal{Y}_M$, let $m^f(r)$ be the M valued function on $\mathbb{R}^+$ as defined in 4.5 . If

i) For each $\xi \in \mathcal{H}$, $m^f(r) v_r \xi \in D_A$ a.e. (depending on ξ)

ii) There exists $g \in \mathcal{Y}_M$, such that for each $\xi \in \mathcal{H}$,

$g(s)\xi = u_s A \, m^f(s) \, v_s \xi$ a.e. (depending on f, ξ)

then we say $f \in D(\alpha\sigma_{-(\cdot)}\beta\theta)$ and put $(\alpha\sigma_{-(\cdot)}\beta\theta)f = g.$

We have the following comments to make regarding the system described in 4.7.

REMARKS 4.8.

a) We can similarly define $(\alpha\sigma_{+(\cdot)}\beta)\theta$ on $\mathcal{Y}_M$.

b) Suppose $f \in \mathcal{Y}_M$ is in $D(\alpha\sigma_{-(\cdot)}\beta\theta)$.

Take $\xi, \eta \in \mathcal{H}$, and define $\phi \in M_*$ by $\phi m = \langle m\xi, \eta\rangle$ $m \in M$.

Then if $C = \alpha\sigma_{-(\cdot)}\beta\theta$ and m^f is as defined in (4.5)

$$\langle (Cf)(r)\xi, \eta\rangle = \langle u_r A m^f(r) v_r\xi, \eta\rangle \quad \text{a.e.} \quad (\text{depending on } \xi).$$

Take $Q_n \in M$, such that $Q_n \to 1$ $\sigma(M,M_*)$ and AQ_n is bounded for each n, and if $\xi \in D_A$ then $AQ_n\xi \to A\xi$.

Then if $(\Gamma_r)_*\phi \in D_{\alpha_*}$ a.e., we have

$$\begin{aligned}\langle (Cf)(r)\xi, \eta\rangle &= \lim_{n\to\infty} \langle u_r[AQ_n]m^f(r)\, v_r\xi, \eta\rangle \\ &= \lim_{n\to\infty} \langle AQ_n\, m^f(r), (\Gamma_r)_*\phi\rangle \\ &= \lim_{n\to\infty} \langle Q_n\, m^f(r), \alpha_*(\Gamma_r)_*\phi\rangle \\ &= \langle m^f(r), \alpha_*(\Gamma_r)_*\phi\rangle \quad \text{a.e. (depending on } \phi \in M_*)\end{aligned}$$

c) It is clear that there exists more than one sequence of projections P_n in M such that

i) $P_n \to 1$ $\sigma(M,M_*)$

ii) P_nA is bounded for each n.

Then if $f \in \mathcal{Y}_M$, we define

$$f_n(s)\xi = \Gamma_s(P_n)\,[f(t)]\xi \quad \text{a.e.}$$

Then $f_n \in \mathcal{Y}_M$, and if $z = \Sigma\, \xi_i \otimes \bar{h}_i \in \mathcal{Z}$, where $\xi_i \in \mathcal{H}$, $h_i \in L^p(\mathbb{R}^+;\mathcal{H})$ and $\Sigma\|\xi_i\| \,.\, \|h_i\| < \infty$, we have

$$\sum_i |\langle f_n(t)\xi_i \, , \, h_i(t)\rangle| \leq \sum_i (M_o)^2 \, \| f(t)\xi_i \| \, . \, \| h_i(t) \| \quad .$$

Also as $n \to \infty$

$$\sum_{i=1}^{\infty} \langle f_n(t)\xi_i \, , \, h_i(t) \rangle \quad \to \quad \sum_{i=1}^{\infty} \langle f(t)\xi_i \, , \, h_i(t) \rangle \, ,$$

by i) and the $\sigma(M,M_*)$ continuity of Γ_t .

It follows by dominated convergence that $f_n = \Gamma_{(\cdot)}(P_n)r \to f \; \sigma(\mathcal{Y}_M, \mathcal{Z}_M)$ as $n \to \infty$.

We will use the sequences P_n, Q_n extensively in the proof of our perturbation theorem.

d) If $f \in \mathcal{Y}_M$, with f(s) in M a.e., then

$$\langle f, Z_r(\phi)\rangle \quad = \quad \int_0^\infty \langle f(s), \; [Z_r(\phi)] \, (s)\rangle \; ds$$

$$= \quad \int_0^\infty \langle f(s) \; , \; (\Gamma_{-s})_* \; \beta_* \; e^{-iT(r-s)}\phi \rangle \; ds$$

for all $\phi \in M_*$, $r \geq 0$.

5. *Main results.*

We recall that if $\omega_o \geq 0$, and if $T \in \mathcal{C}_+(X)$, where X is a Banach space, that T is of type ω_o means that for all $\varepsilon > 0$, that

$$\lim_{t \to \infty} \quad e^{-(\omega_o+\varepsilon)t} \, \| e^{-itT} \| \quad = \quad 0 \quad , \quad [3].$$

We can now state our perturbation theorem (c.f. [5, Theorem 3.1.]).

THEOREM 5.1.

Let M *be a type* I *von Neumann algebra with discrete centre acting on a separable hilbert space* $\mathcal{H}$, *as in* 3.10. *Let* σ_t *be a weakly continuous one parameter group on* M, *with* iT *denoting the infinitesimal generator* *of* $(\sigma_t)_*$, *a strongly continuous group of type* $\omega_o \geq 0$. *Let* Γ_t *be an auxiliary weakly continuous one-parameter group on* M *satisfying* 4.1. $A \in \mathcal{C}(\mathcal{L})$ *is affiliated with* M, *and gives a left multiplication operator* $\alpha = \alpha_A$, *as defined in* 4.2. β *is a linear operator on* M *which is closed and densely defined for the ultraweak topology. Suppose that*

(a) α *is* (σ,Γ,q) *smooth with associated functions* Y, $\hat{Y}$ *in* $B(M, \mathcal{Y}_M)$, *such that* $u \mapsto Yu$, $\hat{Y}u$ *are continuous for the* $\sigma(M,M_*)$ *and* $\sigma(\mathcal{Y}_M, \mathcal{Z}_M)$ *topologies. Moreover assume that if* $\phi \in M_*$, *then* $(\Gamma_t)_*\phi \in D_{\alpha_*}$ a.e. *in* $\mathbb{R}$.

(b) β *is* $(\sigma,\Gamma,p)_*$ *smooth with associated maps* Z_t, $\hat{Z}_t$, *and for each* $\phi \in M_*$, $t \to \langle f, Z_t(\phi)\rangle$, $\langle f, Z_t(\phi)\rangle$ *are continuous from the right on* $[0,\infty)$. *Moreover Graph* β *is countably* σ*-weakly generated.*

(c) $\alpha\sigma_{\pm(\cdot)}\beta\theta$ *belong to* $B(\mathcal{Y}_M)$, *and are* $\sigma(\mathcal{Y}_M, \mathcal{Z}_M)$ *continuous with common bound* $\leq N$.

Then for each k *in* $D = \{\lambda \in \mathbb{C} : |\lambda| < 1/N\}$, *there is a weakly continuous one-parameter group* σ_t^k *on* M, *with* $iT(k)$ *the infinitesimal generator of the strongly continuous group* $(\sigma_t^k)_*$, *of type* ω_o, *uniquely determined by* (σ,α,β) *with the following properties:*

1) α *is* (σ^k,Γ,q) *smooth, with associated functions* Y^k, $\hat{Y}^k$ *in* $B(M, \mathcal{Y}_M)$ *and* $|\alpha|_{(\sigma^k,\Gamma,q)} \leq (1 - |k|N)^{-1}|\alpha|_{(\sigma,\Gamma,q)}$.

2) β_* *is* $(\sigma^k,\Gamma,p)_*$ *smooth with associated functions* Z_r^k, $\hat{Z}_r^k$ *in* $B(M_*, \mathcal{Z}_M)$ *for each* $r \geq 0$, *with*

$$|\beta_*|_{(\sigma^k,\Gamma,p)_*} \leq (1 - |k|N)^{-1}|\beta_*|_{(\sigma,\Gamma,p)_*} .$$

3) *For each* k *in* D, *there exists* Z^k *in* $B(M_*, \mathcal{Z}_M)$ *with*

$$[Z^k\phi](s) = (\Gamma_s)_* \beta_* e^{-iT(k)s}\phi \quad \text{a.e. in } \mathbb{R}^+, \text{ for each } \phi \in M_* ,$$

and $\|Z^k\| \leq M_o^2 (1 - |k|N)^{-1}|\beta_*|_{(\sigma,\Gamma,p)_*}$.

Similarly for the other half line, with $\hat{Z}$ *in* $B(M_*, \mathcal{Z}_M)$.

4) $iT(k)$ *is similar to* iT. *More specifically, there exists for each* k *in* D, *two non singular weakly continuous operators* $W_\pm(k)$ *on* M *given by*

$$\langle W_+(k)m,\phi\rangle = \langle m,\phi\rangle + ik\langle Y(u), \hat{Z}^k(\phi)\rangle$$

$$\langle W_-(k)m,\phi\rangle = \langle m,\phi\rangle - ik\langle \hat{Y}(u), Z^k(\phi)\rangle$$

for all m *in* M, ϕ *in* M_*, *and satisfying*

$$T(k) = W_\pm(k)_*^{-1}\ T\ W_\pm(k)_* \quad .$$

Moreover $W_\pm(k) = \sigma\text{-weak}\lim_{t\to\pm\infty} \sigma_t\ \sigma^k_{-t}$

5) $T(k) \supseteq T + k\alpha_*\beta_*$; $T \supseteq T(k) - k\alpha_*\beta_*$.

PROOF.

To construct the perturbed group, we first derive two semigroups.

Let $U_o(t) = \sigma_{-t}$ be the unperturbed group. For each fixed m in M, set

$$S_o(t)m = u_t A\sigma_{-t}(m)v_t \qquad \text{for } t \geq o.$$

By our smoothness assumptions, $S_o(\cdot)m \in \mathcal{Y}_M$, $S_o(\cdot)m = Y(m)$, and

$$\| S_o(\cdot)m\| \leq |\alpha|\ \|m\| \qquad m \in M.$$

Inductively define for $n \geq 1$,

$$S_n(\cdot)m = (\alpha\sigma_{-(\cdot)}\beta\theta)\ S_{n-1}(\cdot)m \quad , \quad m \in M.$$

Then $S_n(\cdot)m \in \mathcal{Y}_M$ and $\| S_n(\cdot)m\| \leq N^n|\alpha|\ \|m\|$.

To construct our perturbed semigroups, we first define operators $U_n(t)$ as follows:

For $o < t < \infty$, $n \geq 1$, consider for $m \in M$, $\phi \in M_*$,

$$J_n(t,m,\phi) = \langle S_{n-1}(\cdot)m,\ Z_t(\phi)\rangle \ .$$

Then $|J_n(t,m,\phi)| \leq N^{n-1} \|m\| \, |\alpha| \, |\beta_*| \, \|\phi\|$

and $\phi \mapsto J_n(t,m,\phi)$ is linear on M_* . Hence there exists an unique w_t in M

such that $\langle w_t, \phi \rangle = J_n(t,m,\phi)$.

Define $U_n(t)m = w_t$. Set $U_n(o) = o$ $(n \geq 1)$.

Now we define

$$U(t,k)m = \sum_{n=o}^{\infty} (-ik)^n \, U_n(t)m \quad , \quad \text{for all } m \text{ in } M, \ k \text{ in } D.$$

We will show that $\{U(t,k)\}_{o \leq t < \infty}$ is a weakly continuous semigroup on M.

LEMMA 5.2.

The family U(t,k) *is a semigroup of operators in* B(M) *for each* k *in* D.

PROOF.

Let us consider the series $\sum_{n=1}^{\infty} (-ik)^n \, U_n(t)$.

For each u in M, we have

$$\left\| \sum_{n=1}^{\infty} (-ik)^n \, U_n(t)u \right\| \leq \sum_{n=1}^{\infty} \|U_n(t)u\| \, |\kappa|^n \leq \sum |\kappa|^n \, N^{n-1} |\alpha| \, |\beta_*| \, \|u\| .$$

Thus for any k in D, the series converges absolutely and uniformly with respect to t, $(o \leq t < \infty)$.

Since $U(t,k) = U_o(t) + \sum_{n=1}^{\infty} (-ik)^n \, U_n(t)$

it follows that U(t,k) is in B(M) for each $t \geq o$.

We now observe that if $\xi \in \mathcal{H}$ then

$$U_n(t)(u) \, v_t\xi \in D_A \quad , \text{ a.e. in } \mathbb{R}^+,$$

and $u_t A \, U_n(t)(u) \, v_t\xi = S_n(t)(u)\xi$, a.e. in $\mathbb{R}^+$,

since, in the notation of (4.5), if we take $f = S_{n-1}(u)$, then

$m^f(t) = U_n(t)u$ in $\mathbb{R}^+$. Thus if $m \in M$, $\phi \in M_*$,

$$\langle U_n(t)u, \varphi\rangle = \langle S_{n-1}(\cdot)u\ ,\ Z_t(\phi)\rangle$$
$$= \lim_{j\to\infty} \langle \Gamma_{(\cdot)}(P_j)\ S_{n-1}(\cdot)u,\ Z_t(\phi)\rangle$$
$$= \lim_{j\to\infty} \sum_i \int_0^\infty \langle \Gamma_t(P_j)\left[u_t A U_{n-1}(t)uv_t\right]\xi_i\ ,\ h_i(t)\rangle\ dt\ ,$$

if $Z_t(\phi) = i_M^*\ \left[\sum \xi_i \otimes \bar{h}_i\right]$, where $\xi_i \in \mathcal{H}$, $h_i \in L^p(\mathbb{R}^+;\mathcal{H})$ and $\sum \|\xi_i\|\ \|h_i\| < \infty$.

Thus $\langle U_n(t)u, \varphi\rangle = \lim_{j\to\infty} \sum \int_0^\infty \langle u_r\ \Gamma_j\ A\ U_{n-1}(r)(u)\ v_r\ \xi_i\ ,\ h_i(r)\rangle\ dr$

$$= \lim_{j\to\infty} \int_0^\infty \langle \Gamma_r\left(P_j A U_{n-1}(r)(u)\right)\ ,\left[Z_t(\phi)\right](r)\rangle\ dr$$
$$= \lim_{j\to\infty} \int_0^t \langle P_j A U_{n-1}(r)u,\ \beta_* e^{-iT(t-r)}\phi\rangle\ dr\ .$$

Now in order to prove that U(t,k) is a semigroup, it is enough to show that

$$U_n(t+s) = \sum_{j=o}^{n} U_j(t)\ U_{n-j}(s)\ ,\quad \text{for } s,\ t \geq o.$$

We do this by induction on n. Suppose $n \geq 1$. Then if $u \in M$, $\phi \in M_*$,

$$\sum_{j=o}^{n} \langle U_j(t)\ U_{n-j}(s)u,\ \phi\rangle$$
$$= \lim_{m\to\infty} \sum_{j=1}^{n} \int_0^t \langle P_m A U_{j-1}(r) U_{n-j}(s)u,\ \beta_* e^{-iT(t-r)}\phi\rangle\ dr\ +\ \langle U_o(t)U_n(s)u, \phi\rangle$$
$$= \lim_{m\to\infty} \int_0^t \langle P_m A\ U_{n-1}(r+s)u,\ \beta_*\ e^{-iT(t-r)}\phi\rangle\ dr\ +\ \langle U_o(t)U_n(s)u, \phi\rangle$$

by our induction hypothesis

$$= \lim_{m\to\infty} \int_s^{s+t} \langle P_m A\ U_{n-1}(w)u,\ \beta_*\ e^{-iT(t+s-w)}\phi\rangle\ dw\ +\ \langle U_o(t)U_n(s)u, \phi\rangle$$
$$= \langle U_n(t+s)u, \phi\rangle\ .$$

The lemma follows.

LEMMA 5.3.

The semigroup $\{U(t,k) : t \geq o\}$ *is a weakly continuous semigroup of operators of* M*, of type* ω_o .

PROOF.

For $n \geq 1$, $u \in M$, $\phi \in M_*$, $t \geq o$

$$\langle U_n(t)u,\phi\rangle = \langle S_n(\cdot)u, Z_t(\phi)\rangle \quad .$$

Now $u \to S_o u = Y(u)$ is $\sigma(\mathcal{Y}_M, \mathcal{Z}_M) - \sigma(M,M_*)$ continuous.

Moreover $\alpha\sigma_{-(\cdot)}\beta\theta$ is weakly continuous.

Hence $u \to \dot{S}_n u$ is weakly continuous, which shows that $u \to \langle U_n(t)u, \phi\rangle$ is $\sigma(M,M_*)$ continuous.

It follows that $u \mapsto U(t,k)u$ is $\sigma(M,M_*)$ continuous. Moreover, by assumption, for each fixed ϕ in M_*, $t \mapsto Z_t(\phi)$ is $\sigma(\mathcal{Z}_m, \mathcal{Y}_M)$ continuous from the right, so that $t \mapsto U_n(t)u$ is $\sigma(M,M_*)$ continuous from the right. Hence, being the sum of $U_o(t)u$ and $\sum_{n=1}^{\infty} (-ik)^n U_n(t)u$, (which converges uniformly and absolutely with respect to t), $U(t,k)u$ is $\sigma(M,M_*)$ continuous from the right. That is, $U(t,k)_*$ is weakly $(\sigma(M_*,M))$ continuous from the right. Hence by [7,p.233], $U(t,k)_*$ is strongly continuous in t. The last claim is clear.

We denote the infinitesimal generator of $U(t,k)_*$ by $-iT(k)$, for k in D, and write σ^k_{-t} for $U(t,k)$, $t \geq o$.

REMARK 5.4.

We may replace T by -T, and define for $t \geq o$, $\hat{U}_n(t)$ by

$$\hat{U}_o(t) = \sigma_t ,$$

$$\langle \hat{U}_n(t)u,\phi\rangle = -\langle S_{n-1}(\cdot)u, \hat{Z}_t(\phi)\rangle \quad u \in M,\ \phi \in M_*$$

where $\hat{S}_o(\cdot)u = \hat{Y}(u)$, and $\hat{S}_n(\cdot)u = (\alpha\sigma_{+(\cdot)}\beta\theta)\hat{S}_{n-1}(\cdot)u$.

Then if we put

$$\hat{U}(t,k) = \sum_{n=o}^{\infty} (ik)^n \hat{U}_n(t)$$

we deduce the existence of a weakly continuous semigroup on M, for each k in D. We shall denote by $i\hat{T}(k)$, the infinitesimal generator of the strongly continuous semigroup $\hat{U}(t,k)_*$ on M_* of type ω_o, and for $\hat{U}(t,k)$ we write $\hat{\sigma}^k_t$, $t \geq o$.

LEMMA 5.5.

For all k *in* D, $t \geq o$,

$$\hat{\sigma}^k_t \; \sigma^k_{-t} = 1 = \sigma^k_{-t} \; \hat{\sigma}^k_t .$$

PROOF.

We will inductively show, for $n \geq 1$ that $\sum_{j=o}^{n} \hat{U}_j(t)\, U_{n-j}(t) = o$.

Take $u \in M$, $\phi \in M_*$; now

$$\sum_{j=o}^{n} \langle \hat{U}_j(t)\, U_{n-j}(t)u, \phi \rangle$$

$$= - \sum_{j=1}^{n} \lim_{m\to\infty} \int_o^t \langle (P_m\, A)\hat{U}_{j-1}(r)\, U_{n-j}(t)u,\ \beta_*\, e^{iT(t-r)}\phi \rangle\, dr \; + \langle \hat{U}_o(t) U_n(t)u, \phi \rangle$$

$$= - \sum_{l+k+s=n-1} \lim_{m\to\infty} \int_o^t \langle (P_m\, A)\hat{U}_l(r) U_k(r) U_s(t-r)u,\ \beta_* e^{+iT(t-r)}\phi \rangle\, dr$$

$$+ \langle \hat{U}_o(t) U_n(t)u, \phi \rangle$$

$$= - \lim_{m\to\infty} \int_o^t \langle (P_m A)\, U_{n-1}(t-r)u,\ \beta_*\, e^{iT(t-r)}\phi \rangle\, dr \quad + \langle \hat{U}_o(t) U_n(t)u, \phi \rangle$$

by our induction hypothesis

$= 0.$

Hence $\hat{\sigma}^k_t\, \sigma^k_{-t} = 1$, and similarly for the other relation.

As a corollary we see that $T(k) = \hat{T}(k)$ for all k in D, and we have a one-parameter weakly continuous group σ^k_t, for $k \in D$.

LEMMA 5.6.

α *is* (σ^k,Γ,q) *smooth with* $|\alpha|_{(\sigma^k,\Gamma,q)} \leq (1 - |\kappa|N)^{-1} |\alpha|_{(\sigma,\Gamma,q)}$.

PROOF.

If $u \in M$, $\xi \in \mathcal{H}$, then $U_n(t)(u)\, v_t\xi \in D_A$ a.e. in $\mathbb{R}^+$ and

$$S_n(t)u\,\xi \;=\; u_t A\, U_n(t)(u)\, v_t\xi \quad \text{a.e.}$$

Let $$Y_n(u) \;=\; \sum_{m=o}^{n} (-ik)^m S_m u.$$

It follows that for k in D,

$$\sum \| (-ik)^m S_m u\| \le \sum |\kappa|^m N^m |\alpha|_{(\sigma,\Gamma,q,+)} \|u\| = (1 - |\kappa|N)^{-1} |\alpha|_{(\sigma,\Gamma,q,+)} \|u\| .$$

Hence there exists Y^k in $B(M, \mathcal{Y}_M)$ such that

$$Y^k(u) \;=\; \sum_{m=o}^{\infty} (-ik)^m S_m u \quad , \quad u \in M.$$

Fix $u \in M$, $\xi \in \mathcal{H}$. Then we can pick a subsequence n_j such that

$$[Y_{n_j}(u)](s)\xi \;\rightarrow\; [Y^k(u)](s)\xi \quad , \text{ a.e. in } \mathbb{R}^+ .$$

Note $$[Y_n(u)](s) \;=\; \sum_{m=o}^{n} u_t\, A\, U_m(t)(u)\, v_t\xi \;, \text{ a.e. in } \mathbb{R}^+ .$$

But $\sum_{m=1}^{\infty} (-ik)^m U_m(t)u$ converges uniformly in t. It follows, since $u_t A$ is closed, that $\sum (-ik)^m U_m(t)\, v_t\xi \in D_A$ a.e. , and

$$u_t\, A\, (\sum(-ik)^m U_m(t)(u)\, v_t\xi) = \lim_j [Y_{n_j}(u)](t)\xi \;=\; [Y^k(u)]\,(t) \quad \text{a.e.}$$

In other words $\sigma^k_{-t}(u) v_t\xi \;\in\; D_A$ a.e.

and $[Y^k(u)]\,(t)\xi \;=\; u_t\, A\, \sigma^k_{-t}(u) v_t\xi$, a.e. in $\mathbb{R}^+$.

Similarly for the other half line.

REMARK 5.7.

We now investigate the $\sigma(\mathcal{Y}_M, \mathcal{Z}_M)$ continuous operator $C = \alpha\sigma_{-(\cdot)}\beta\theta$ in $B(\mathcal{Y}_M)$. We denote the restriction of C^* to $\mathcal{Z}_m$ by C_* (as usual). Take v in D_β, and [a,b] a compact interval in $\mathbb{R}^+$. Define f in $\mathcal{Y}_M$ by

$f(s) = \Gamma_s(v)\, \chi_{(a,b)}(s)$.

Take any ξ, η in $\mathcal{H}$, and define $\phi \in M_*$ by $\phi(m) = \langle m\xi,\eta\rangle$ for $m \in M$. Then for almost all s in $\mathbb{R}^+$, depending on ϕ, we have

$$\langle (Cf)(s)\xi,\eta\rangle = \int_0^s \langle f(\tau), \Gamma_{-\tau}^1 \beta_* e^{-i(s-\tau)T}\alpha_*(\Gamma_s^1)_*\phi\rangle \chi_{(a,b)}(\tau)\, d\tau$$

$$= \int_0^s \langle v, \beta_* e^{-i(s-\tau)T}\alpha_*(\Gamma_s^1)_*\phi\rangle \chi_{(a,b)}(\tau)\, d\tau$$

$$= \begin{cases} 0 & \text{if } s < a \\ \int_a^s \langle \sigma_{-(s-\tau)}\beta(v), \alpha_*(\Gamma_s^1)_*\phi\rangle\, d\tau & \text{if } s \in [a,b) \\ \int_a^b \langle \sigma_{-(s-\tau)}\beta(v), \alpha_*(\Gamma_s^1)_*\phi\rangle\, d\tau & \text{if } s \geq b. \end{cases}$$

For $(s,t) \in (\mathbb{R})^2$, define for $h \in L^p(\mathbb{R}^+)$,

$$u(s,t) = \langle \sigma_{-(s-t)}\beta(v), \alpha_*(\Gamma_s^1)_*\phi\rangle\, h(s),$$

which is jointly measurable. Moreover

$$\int_a^b\int_\tau^\infty |u(s,\tau)|\, ds\, d\tau \leq \int_a^b\left(\int_\tau^\infty |\langle u_s A\, \sigma_{-(s-\tau)}(\beta(v))v_s\xi,\eta\rangle h(s)|\, ds\right) d\tau$$

$$\leq \int_a^b \|\sigma_\tau \beta(v)\|\, \|\xi\|\, \|\eta\|\, \|h\|\, d\tau < \infty$$

as $[a,b]$ is compact, and for each fixed τ, we have a.e.

$$\langle u_s A\, \sigma_{-(s-\tau)}\beta(v) v_s\xi,\eta\rangle h(s) = \lim_{n\to\infty} \langle u_s(AQ_n)\sigma_{-(s-\tau)}\beta(v)v_s\xi,\eta\rangle h(s)$$

$$= \lim_{n\to\infty} \langle Q_n\sigma_{-(s-\tau)}\beta(v), \alpha_*(\Gamma_s^1)_*\phi\rangle h(s)$$

$$= \langle \sigma_{-(s-\tau)}\beta(v), \alpha_*(\Gamma_s^1)_*\phi\rangle h(s).$$

Hence we can apply Fubini's theorem, and deduce that if

$$g = i_M^*(\xi \otimes (\bar\eta h)) \in \mathcal{Z}_M, \quad \text{that}$$

$$\langle Cf, g\rangle = \int_a^b \langle Cf(s)\xi, \eta\rangle h(s)\, ds$$

$$= \int_a^b \left(\int_a^s \langle \sigma_{-(s-\tau)}\beta(v), \alpha_*(\Gamma_s^1)_*\phi\rangle\, d\tau\right) h(s)\, ds$$

$$+ \int_b^\infty \left(\int_a^b \langle \sigma_{-(s-\tau)}\beta(v), \alpha_*(\Gamma_s^1)_*\phi\rangle\, d\tau\right) h(s)\, ds$$

$$= \int_a^b\int_\tau^\infty \langle \sigma_{-(s-\tau)}\beta(v), \alpha_*(\Gamma_s^1)_*\phi\rangle h(s)\, ds\, d\tau$$

It follows that whenever $g \in \mathcal{Z}_M$, $g = i_M^*(z)$ where $z = \sum_{i=1}^{\infty} \xi_i \otimes \bar{k}_i$

$\xi_i \in \mathcal{H}$, $k_i \in L^p(\mathbb{R}^+;\mathcal{H})$, and $\sum \|\xi_i\| \cdot \|k_i\| < \infty$, that

$$\langle Cf,g\rangle = \int_a^b \left(\int_\tau^\infty \langle u_s A \sigma_{-(s-\tau)}\beta(v) v_s \xi_i, k_i(s)\rangle ds \right) d\tau .$$

On the other hand

$$\langle Cf,g\rangle = \langle f,C_*g\rangle = \int_a^b \langle \Gamma_\kappa(v), [C_*g](\tau)\rangle d\tau .$$

Thus almost everywhere, (depending on $v \in D_\beta$)

$$\langle \Gamma_\tau(v), [C_*g](\tau)\rangle = \sum_i \int_\tau^\infty \langle u_s A \sigma_{-(s-\tau)}(\beta(v))v_s\xi_i, k_i(s)\rangle ds .$$

LEMMA 5.8.

Fix $t > o$, *and define for* $\phi \in M_*$, $m \in \mathbb{Z}$,

$$U_{m,t}(r)_*\phi = \begin{cases} U_m(t-r)_*\phi & \text{if } r \in [o,t] \\ 0 & \text{if } r > t. \end{cases}$$

Then $U_{m,t}(r)_*\phi \in D_{\beta_*}$ a.e. *and*

$$\Gamma_{-r}\, \beta_*\, U_{m,t}(r)_*\phi = [(C_*)^m Z_t(\phi)](r) \quad \text{a.e.} \quad \textit{(depending on } \phi \textit{ and } t).$$

PROOF.

For $m = o$, see the definition of $Z_t(\phi)$.

$m \geq 1$. Let $w \in D_\beta$. Then

$$\langle \Gamma_r(w), [C_*^m Z_t(\phi)](r)\rangle = \langle \Gamma_r(w), [C_*\, C_*^{m-1} Z_t(\phi)](r)\rangle$$

$$= \lim_{p\to\infty} \int_r^\infty \langle \Gamma_s[(P_pA)\sigma_{-(s-r)}\beta(w)], [C_*^{m-1} Z_t(\phi)](s)\rangle \; ds$$

$$= \lim_{p\to\infty} \int_o^\infty \langle \Gamma_{s+r}[P_pA\; \sigma_{-s}\beta(w)], \; [C_*^{m-1} Z_t(\phi)](s+r)\rangle \; ds.$$

Now
$$[C_*^{m-1} Z_t(\phi)](s+r) = (\Gamma_{-s-r})_*\, \beta_*\, U_{m-1,t}(s+r)_*\phi$$
$$= (\Gamma_{-s-r})_*\, \beta_*\, U_{m-1,t-r}(s)_*\phi$$
$$= (\Gamma_{-r})_*\, [C_*^{m-1} Z_{t-r}(\phi)](s) \quad \text{a.e.}$$

if $r \leq t$, and zero otherwise.

Hence
$$\langle \Gamma^1_r(w), [C_*^m Z_t(\phi)](r)\rangle = \lim_{p\to\infty} \int_0^\infty \langle \Gamma^1_s[\Gamma_p A\, \sigma_{-s}(\beta(w))], [C_*^{m-1} Z_{t-r}(\phi)](s)\rangle\, ds$$
$$= \langle S_o\beta(w),\ C_*^{m-1} Z_{t-r}(\phi)\rangle$$
$$= \langle S_{m-1}\beta(w),\ Z_{t-r}(\phi)\rangle$$
$$= \langle U_m(t-r)\beta(w), \phi\rangle ,$$

a.e. (depending on t,w,ϕ).

But Graph β is ultraweakly countably generated. Hence for all w in D_β ,

$$\langle \Gamma_r(w), [C_*^m Z_t(\phi)](r)\rangle = \langle U_m(t-r)\beta w, \phi\rangle$$

a.e. (depending on $t > o$ and ϕ in M_*).

Hence $U_m(t-r)_*\phi \in D_{\beta_*}$ a.e., and

$$(\Gamma_r)_* [C_*^m Z_t(\phi)](r) = \beta_* U_{m,t}(r)_*\phi \qquad \text{a.e.}$$

LEMMA 5.9.

(a) β_* *is* $(\sigma^k,\Gamma,p)_*$ *smooth, with associated functions* Z_r^k , $r \geq o$, *and*

$$|\beta_*|_{(\sigma^k,\Gamma,p)_*} \leq (1 - |k|N)^{-1} |\beta_*|_{(\sigma,\Gamma,p)_*} .$$

(b) *There exists* Z^k *in* $B(M_*, \mathcal{Z}_M)$ *satisfying, for* ϕ *in* M_*,

$$(Z^k\phi)(s) = (\Gamma_s)_* \beta_*, e^{-iT(k)s}\phi \qquad \text{a.e.}$$

and $$\| Z^k\| \leq M_o^2 (1 - |k|N)^{-1} |\beta_*|_{(\sigma,\Gamma,p)_*} .$$

Similarly for the other half line, with associated maps $\hat{Z}_r$, $r \geq o$, *and* $\hat{Z}$ *in* $B(M_*, \mathcal{Z}_M)$.

PROOF.

(a) $\sum \| (-ik)^m C_*^m Z_t(\phi)\| \leq |k|^m N^m |\beta_*| \, \|\phi\|$ for each ϕ in M_*, $t \geq o$.

Hence if $k \in D$, the series converges uniformly and absolutely in t to $Z_t^k(\phi)$ say. By Remark 3.7, we see that for each fixed $t > 0$, $\phi \in M_*$, there is a subsequence n_k such that

$$[Z_t^k(\phi)](s) = \lim_{j\to\infty} \sum_{m=1}^{n_j} (-ik)^m [C_*^m Z_t(\phi)](s) \quad \text{a.e.}$$

Let W be a null set in $\mathbb{R}^+$, such that if $s \notin W$.

i) $$[Z_t^k(\phi)](s) = \lim_{j\to\infty} \sum_{m=1}^{n_j} (-ik)^m [C_*^m Z_t(\phi)](s)$$

ii) $$[C_*^m Z_t(\phi)](s) = (\Gamma_{-s})_* U_{m,t}(s)_* \quad , \quad m = 0, 1, 2, \ldots\ldots$$

Take $s \notin W$, and b satisfying $\Gamma_{-s}^{-1}(b) \in D_\beta$.

Then
$$\langle [Z_t(\varphi)](s), b\rangle = \lim_{j\to\infty} \sum_{m=1}^{n_j} \langle [C_*^m Z_t(\varphi)](s), b\rangle (-i\kappa)^m$$
$$= \lim_{j\to\infty} \sum_{m=1}^{n_j} \langle (\Gamma_{-s}^{-1})_* \beta_* U_{m,t}(s)_*\varphi, b\rangle (-i\kappa)^m$$
$$= \lim_{j\to\infty} \sum_{m=1}^{n_j} \langle U_m(t-s)_*\varphi, \beta\Gamma_{-s}^{-1}(b)\rangle (-i\kappa)^m \chi_{[o,t]}(s)$$
$$= \langle e^{-iT(\kappa)(t-s)}\varphi, \beta\Gamma_{-s}^{-1}(b)\rangle \chi_{[o,t]}(s)$$

It follows that $e^{-iT(\kappa)(t-s)}\varphi \in D_\beta$ for $s \in [o,t]\cap \complement W$, and

$$[Z_t^\kappa(\varphi)](s) = (\Gamma_{-s}^{-1})_* \beta_* e^{-iT(\kappa)(t-s)}\varphi \quad \text{a.e.}$$

b) From a) it follows that if $\phi \in M_*$, k in D, that there exists z_n in $\mathcal{Z}_M$ such that

$$z_n(s) = (\Gamma_{+s})_* \beta_* e^{-iT(k)s}\phi\, \chi_{[o,n]}(s) \quad \text{a.e.}$$

and $$\| z_n \| \le M_o^2 (1 - |k|N)^{-1} |\beta_*| \, \|\phi\| .$$

The result follows from 3.10.

We can now easily deduce the following:

LEMMA 5.10.

The semigroups σ_{-t} *and* σ_{-t}^k *satisfy for* $t \ge o$ *the following relations if* $u \in M$, $\phi \in M_*$, $k \in D$:

$$\langle \sigma_{-t}^k(u) - \sigma_{-t}(u), \phi\rangle = -i\,k \langle Y^k(u), Z_t(\phi)\rangle$$

$$\langle \sigma_{-t}^k(u) - \sigma_{-t}(u), \phi\rangle = -i\,k \langle Y(u), Z_t^k(\phi)\rangle$$

with analogous results for the other half line.

LEMMA 5.11.

For each k *in* D, $T(k) \supseteq T + k\,\alpha_* \beta_*$

$T \supseteq T(k) - k\,\alpha_* \beta_*$

PROOF.

Let $A = U|A|$ be a polar decomposition of A, and Q_n (4.7) to be chosen as spectral projections of A. Then take $P_n' = U\,Q_n\,U^*$.

For $z \in C$, with $\text{im } z > \omega_o$, and $k \in D$, let $R(z,k) = (T(k) - z)^{-1}$ and $R(z) = (T - z)^{-1}$.

Then $e^{izt}(-ik)\langle Y^k(u), Z_t(\phi)\rangle \in L^1(o,\infty)$, for $u \in M$, $\phi \in M_*$; and by dominated convergence:

$$i\int_0^\infty e^{izt}(-i\kappa)\langle Y^\kappa(u), Z_t(\phi)\rangle\, dt$$

$$= i\int \lim_{n\to\infty} (-i\kappa)\, e^{izt}\langle \Gamma_{(\cdot)}(P_n')\, Y^\kappa(u), Z_t(\phi)\rangle\, dt$$

$$= i\int_0^\infty \lim_{n\to\infty} e^{izt}(-i\kappa)\int_0^t \langle \Gamma_s(P_n'A\,\sigma_{-s}(u)), [Z_t(\phi)](s)\rangle\, ds\, dt$$

$$= i\lim_{n\to\infty}\int_0^\infty\int_0^t e^{izt}(-i\kappa)\langle \Gamma_s(P_n'A\,\sigma_{-s}(u)), [Z_t(\phi)](s)\rangle\, ds\, dt$$

$$= i\lim_{n\to\infty}\int_0^\infty\int_0^\infty e^{iz(s+r)}(-i\kappa)\langle P_n'A\,\sigma_{-s}(u), \beta_*\, e^{-irT}\phi\rangle\, dr\, ds,$$

where we have used Fubini's theorem, the use of which is easily justified as in Remark 5.7.

Now suppose $R(z)\phi \in D_{\alpha_*\beta_*}$. Now by 3.7 and 5.9, for each ψ in M_*,

$$r \mapsto \beta_*\, e^{-irT}\psi \in L^p(\mathbb{R}^+; M_*).$$

Hence for all $\psi \in M_*$; $R(z)\psi \in D_{\beta_*}$ and

$$\langle u, \beta_* R(z)\psi\rangle = +i\int_0^\infty \langle u, \beta_*\, c^{-irT}\psi\rangle\, e^{irz}\, dr \quad \text{for all } u \text{ in } M.$$

Hence

$$i \int_0^\infty e^{izt}(-i\kappa)\langle Y^\kappa(u), Z_t(\phi)\rangle \, dt$$
$$= \lim_{n\to\infty} \int_0^\infty (-i\kappa) e^{izs} \langle AQ_n \sigma_{-s}(u), \beta_* R(z)(v)\rangle \, ds$$
$$= -(\kappa) \langle u, R(z,\kappa)\alpha_* \beta_* R(z)\phi\rangle$$
$$= \langle u, R(z,\kappa)\phi\rangle - \langle u, R(z)\phi\rangle \qquad \text{by taking Laplace transforms.}$$

It follows easily that $T \geq T(k) - k\,\alpha_*\beta_*$.

The other relation is proved similarly.

We now consider the similarity operators.

LEMMA 5.12.

For $u \in M$, $\phi \in M_*$, *the expressions*

$$\langle W_-(k)u,\phi\rangle = \langle u,\phi\rangle - i\,k\,\langle \hat{Y}(u), Z^k(\phi)\rangle$$
$$\langle W_+(k)u,\phi\rangle = \langle u,\phi\rangle + i\,k\,\langle Y(u), \hat{Z}^k(\phi)\rangle$$
$$\langle H_-(k)u,\phi\rangle = \langle u,\phi\rangle + i\,k\,\langle \hat{Y}^k(u), Z(\phi)\rangle$$
$$\langle H_+(k)u,\phi\rangle = \langle u,\phi\rangle - i\,k\,\langle Y^k(u), \hat{Z}(\phi)\rangle \qquad (k \in D)$$

define ultraweakly continuous bounded operators on M. *Moreover*

$$W_-(k)\,H_-(k) = H_-(u)\,W_-(k) = 1$$
$$W_+(k)\,H_+(k) = H_+(k)\,W_+(k) = 1.$$

PROOF.

The right hand sides clearly define bounded linear operators on M.

But $\hat{Y}^k(u) = \sum_{n=0}^{\infty} \hat{S}_n u\,(-ik)^n$.

Using the weak continuity of $\alpha\sigma_{+(\cdot)}\beta\theta$ and $u \mapsto \hat{S}_0 u$, we see that $W_-(k)$ and $H_-(k)$ are ultraweakly continuous. Let us write

$$W_-(k) = 1 + kP, \quad \text{and } H_-(k) = 1 + kQ.$$

Take $u \in M$, $\phi \in M_*$, and we have

$$\langle PQu,\phi\rangle = \langle Qu, P_*\phi\rangle = i\,\langle \hat{Y}^k(u), Z(P_*\phi)\rangle$$
$$= \lim_{n\to\infty} \int_0^\infty i\langle P_n A\,\sigma_s^k(u), \beta_*\,\sigma_{-s}\,P_*\phi\rangle \, ds.$$

For any w in D_β, $\phi \in M_*$ one has

$$\langle w, \beta_* e^{-isT} P_* \phi\rangle = \langle P\, \sigma_{-s}\beta(w), \phi\rangle$$
$$= \lim_{m\to\infty} - \int_0^\infty i \langle P_m A\, \sigma_r\, \sigma_{-s}\, \beta(w), \beta_* e^{-irT(\kappa)}\phi\rangle \, dr$$
$$= -i \lim_{m\to\infty} \int_0^s \; -i \lim_{m\to\infty} \int_s^\infty$$

The two parts are evaluated separately as follows

$$\lim_{m\to\infty} \int_0^s = \lim_{m\to\infty} \int_0^s \langle P_m A\, \sigma_{-(s-r)}\beta w, \beta_* e^{-irT(\kappa)}\phi\rangle \, dr$$
$$= \frac{i}{\kappa} \langle \sigma^\kappa_{-s}\beta w - \sigma_{-s}\beta w, \phi\rangle$$
$$= \frac{i}{\kappa} \langle w, \beta_* e^{-isT(\kappa)}\phi\rangle - \frac{i}{\kappa}\langle w, \beta_* e^{-isT}\phi\rangle \quad ,$$

a.e. using Lemma 5.10.

Also by Remark 5.7,

$$\lim_{m\to\infty} \int_s^\infty \langle P_m A\, \sigma_{r-s}\beta w, \beta_* e^{-irT(k)}\phi\rangle \, dr = \langle \Gamma_{-s}(w), [\hat{C}_* Z^k(\phi)](s)\rangle$$

a.e., depending on k, w, ϕ .

Since Graph β is ultraweakly countably generated, we conclude that

$$\beta_* e^{-isT} P_* \phi = \frac{1}{\kappa}\beta_* e^{-isT(\kappa)}\phi - \frac{1}{\kappa}\beta_* e^{-isT}\phi - i(\Gamma_{-s})_* [\hat{C}_* Z^\kappa(\phi)](s) \quad .$$

Hence
$$\langle PQu, \phi\rangle = \frac{i}{\kappa} \lim_{n\to\infty} \int_0^\infty \langle P_n A\, \sigma_s(u), \beta_* e^{-isT(\kappa)}\phi\rangle \, ds$$
$$- \frac{i}{\kappa} \lim_{n\to\infty} \int \langle P_n A\, \sigma_s(u), \beta_* e^{-isT}\phi\rangle \, ds$$
$$+ \lim_{n\to\infty} \int \langle \Gamma_{-s}(P_n A\, \sigma_s(u)), [C_* Z^\kappa(\phi)](s)\rangle \, ds$$
$$= I_1 + I_2 + I_3 \text{ respectively, say.}$$

$$I_3 = \langle Y^\kappa(u), \hat{C}_* Z^\kappa(\phi)\rangle = \langle \hat{C}\, Y^\kappa(u), Z^\kappa(\phi)\rangle \quad .$$

For $\xi, \eta \in \mathcal{H}$, define $v \in M_*$ by $v(m) = \langle m\xi, \eta\rangle$, $m \in M$.

Then for almost all s, we have by 4.8. b, that

$$\langle [\hat{C}\, Y(u)](s)\xi, \eta\rangle = \lim_{n\to\infty} \int_0^s \langle P_n A\, \sigma_r(u), \beta_* e^{+i(s-r)T_{\alpha_*}} (\Gamma_{-s})_* v\rangle \, dr$$
$$= \frac{1}{ik} \langle \sigma^\kappa_s(u) - \sigma_s(u), \alpha_* (\Gamma_{-s})_* v\rangle$$

by Lemma 5.10.

Hence $\hat{C}\, Y^k(u) = \frac{1}{ik} \cdot \hat{Y}^k(u) - \frac{1}{ik} \cdot \hat{Y}(u)$.

It follows that

$$I_3 = \frac{1}{ik} \langle \hat{Y}^k(u), \; Z^k(\phi) \rangle - \frac{1}{ik} \langle \hat{Y}(u), \; Z^k(\phi) \rangle .$$

The first term of I_3 cancels I_1, and the second term of I_3 is

$$- \frac{1}{k} \langle Pu, \phi \rangle \quad ; \quad \text{while} \quad I_2 = - \frac{1}{k} \langle Qu, \phi \rangle .$$

Hence $\langle PQu, \phi \rangle = - \frac{1}{k} \langle Qu, \phi \rangle - \frac{1}{k} \langle Pu, \phi \rangle$.

This is equivalent to $W_-(k)\, H_-(k) = 1$.

The remainder can be shown in the same manner.

LEMMA 5.13.

With the non singular operators $W_\pm(k)$ *constructed above, we have that*

$$W_\pm(k)\sigma_{-r} = \sigma^k_{-r}\, W_\pm(k) \qquad \textit{for all } k \textit{ in } D, \; r \textit{ in } \mathbb{R} .$$

PROOF.

Consider only $r \geq 0$; and take $u \in M$, $\phi \in M_*$.

Then

$$\langle W_-(\kappa)\sigma_{-r}(u), \phi \rangle = \langle \sigma_{-r}(u), \phi \rangle - \lim_{n\to\infty} \int_0^\infty \langle P_n A\, \sigma_s\, \sigma_{-r}(u), \; \beta_* e^{-isT(\kappa)}\phi \rangle \, ds.$$

The integral on the right hand side can be split into two terms, $\int_0^r$ and $\int_r^\infty$ where,

$$\lim_{n\to\infty} -i\kappa \int_0^r \langle P_n A\, \sigma_s\, \sigma_{-r}(u), \; \beta_* e^{-isT(\kappa)}\phi \rangle \, ds$$

$$= \lim_{n\to\infty} -i\kappa \int_0^r \langle P_n A, \; \sigma_{-(r-s)}(u), \; \beta_* e^{-isT(\kappa)}\phi \rangle \quad ds$$

$$= \langle \sigma^\kappa_{-r}(u), \phi \rangle \quad - \langle \sigma_{-r}(u), \phi \rangle \qquad \text{by Lemma } 5.10;$$

and

$$-i\kappa \lim_{n\to\infty} \int_r^\infty \langle P_{nA}\, \sigma_{-s}\sigma_{-r}(u), \; \beta_* e^{-isT(\kappa)}\phi \rangle \quad ds$$

$$= -i\kappa \lim_{n\to\infty} \int_r^\infty \langle P_n A\, \sigma_{-s}(u), \; \beta_* e^{-isT(\kappa)}\, e^{-irT(\kappa)}\phi \rangle \quad ds .$$

Hence $W_-(\kappa)\sigma_{-r} = \sigma^\kappa_{-r} W_-(\kappa)$.

REMARK 5.14.

However, more than the similarity relations is true. The ultraweak limit $W^{\pm}(k)$ of $\sigma_t^k\,\sigma_{-t}$ as $t \to \pm\infty$ exist and coincide with $W_{\pm}(k)$. To show this, notice that

$$\langle W_+(k)\sigma_{-t}(u),\, e^{itT(k)}\phi\rangle = \langle \sigma_t^k\sigma_{-t}u,\phi\rangle + \lim_{n\to\infty}\int_0^\infty ik\langle P_nA\sigma_{-(t+r)}(u),\beta_* e^{i(t+r)T(k)}\phi\rangle\,dr$$

Hence $\langle W_+(k)(u),\phi\rangle - \langle \sigma_t^k\,\sigma_{-t}(u),\phi\rangle = ik\langle Y(u).\chi_{[t,\infty)},\, Z^k(\phi)\rangle$,

which tends to zero, as $t \to \infty$.

6. *Automorphism groups.*

As seen from our motivating example in §3, if the unperturbed system σ_t is a group of automorphisms, then the resulting perturbation need not be. In this section, we consider how to form groups of *-automorphisms from the perturbations σ_t^k.

Again, M is the von Neumann algebra in 3.10. Suppose σ_t is a weakly continuous one parameter group of *-automorphisms of M, with Γ_t an auxiliary weakly continuous group satisfying (4.1). $B, A \in \mathcal{C}(\mathcal{H})$ are affiliated with M and give left multiplication operators β, α respectively, as defined in 4.2. Suppose $(\sigma,\Gamma,\alpha,\beta)$ satisfy all the hypotheses of Theorem 5.1.

Note that D_β is a right ideal in M, and if $x \in D_\beta$, $y \in M$ then $\beta(xy) = \beta(x).y$. This is the only new property of β that we need.

NOTATION 6.1.

If δ is a linear operator on M, with domain D_δ, we define $\bar{\delta}$ to be the linear map on M, with domain $(D_\delta)^*$ and $\bar{\delta}(x) = \delta(x^*)^*$ for $x \in D_\delta$.

THEOREM 6.2.

a) *If* $k \in D$, $\rho_t^k(x) = \sigma_t^{\bar{k}}(x^*)^*$ *for* $x \in M$, *defines a weakly continuous group on* M, *and if* $iJ(k)$ *denotes the infinitesimal generator of the strongly continuous group* $(\rho_t^k)_*$, *then*

$$J(k) \supseteq T - k(\bar{\alpha})_* \, (\bar{\beta})_*$$

$$T \supseteq J(k) + k(\bar{\alpha})_* \, (\bar{\beta})_*$$

Also $\tilde{W}_\pm(k)(x) = W_\pm(\bar{k})(x^*)^*$ $x \in M$, *defines two weakly continuous invertible operators on* M, *satisfying:*

i). $\tilde{W}_\pm(k)\sigma_t = \rho_t^k \, \tilde{W}_\pm(k)$

ii) $\tilde{W}_\pm(k) = \sigma\text{-weak} \lim\limits_{t \to \pm\infty} \rho_t^k \, \sigma_{-t}$

b) *The following hold for all* x, y *in* M, $t \in \mathbb{R}$, $k \in D$

iii) $\sigma_t^k(xy) = \sigma_t^k(x) \quad \sigma_t(y)$

iv) $\rho_t^k(xy) = \sigma_t(x) \quad \rho_t^k(y)$

v) $W_\pm(k)(xy) = [W_\pm(k)x] \, y$

vi) $\tilde{W}_\pm(k)(xy) = x[\tilde{W}_\pm(k)y]$

c) *Suppose also, in the notation of* (4.5), *that for each* $f \in \mathcal{Y}_M$, $r \mapsto m^f(r)^*$ *is strongly continuous from the left on* $(0,\infty)$. *Then for each* k *in* D, *there is a weakly continuous one parameter group* τ_t^k *on* M, *satisfying:*

vii) $\tau_t^k(xy) = \sigma_t^k(x) \quad \rho_t^k(y)$

Also there are two σ-weakly continuous invertible bounded linear operators $\Omega_\pm(k)$, *for each* k *in* D, *satisfying*

viii) $\Omega_\pm(k)(xy) = W_\pm(k)(x) \, \tilde{W}_\pm(k)(y)$ $\qquad x, y \in M.$

ix) $\Omega_\pm(k)\tau_t^k = \sigma_t \, \Omega_\pm(k)$ $\qquad t \in \mathbb{R}.$

PROOF.

We give the proof of b) and c) only.

b) Using the notation of §5, we show by induction on n that

$$U_n(t)(xy) = U_n(t)(x)\,\sigma_{-t}(y), \qquad x,\ y \in M,\ t \geq o.$$

If $\phi \in M_*$, then

$$\langle U_n(t)(xy),\phi\rangle = \lim_{m\to\infty} \int_o^t \langle P_m A\, U_{n-1}(r)(xy),\ \beta_*\, e^{-i(t-r)T}\phi\rangle\, dr$$

$$= \lim_{m\to\infty} \int_o^t \langle P_m A\, U_{n-1}(r)(x)\,\sigma_{-r}(y),\ \beta_* e^{-i(t-r)T}\phi\rangle\, dr$$

$$= \langle U_n(t)(x)\,\sigma_{-t}(y),\phi\rangle ,$$

using the ideal property of D_β etc. Similarly for $t < o$.

This proves iii), and we can prove v) in a similar fashion. iv) and vi) are immediate consequences of iii) and v).

c) For each k in D, t in $\mathbb{R}$, we can define a linear map τ_t^k on M such that

$$\tau_t^k(x) = \sigma_t^k(x)\ \rho_t^k(1) , \qquad \forall\ x \in M$$

Then

$$\tau_t^k(xy) = \sigma_t^k(x)\ \rho_t^k(y) , \qquad \forall\ x,y \in M,\ t \in \mathbb{R},\ k \in D.$$

Clearly $x \longrightarrow \tau_t^k(x)$ is σ-weakly continuous on M.

The additional continuity that we are now given, shows that $r \mapsto (U_n(r)x)^*$ is strongly continuous from the left on (o, ∞).

But

$$\tau_t^k(x) = \sigma_t^k(x) + \sum_{n=1}^{\infty} \sigma_t^k(1)\, U_n(t)(x^*)^*\, (ik)^n .$$

Hence $\tau_t^k(x)$ is $\sigma(M,M_*)$ continuous in t from the left on (o, ∞). By the group property of τ_t^k, it is continuous from the left on $\mathbb{R}$. Hence $(\tau_t^k)_*$ is a strongly continuous group on $\mathbb{R}$.

The remainder is clear.

REMARKS.

(6.3) For the motivation of the condition in 6.2.b, see the proof of [5, Lemma 4.2.].

(6.4) If, as often happens in practice, it is true that

$$(W_+)_* = \operatorname{st}\lim_{t\to\infty} e^{-itT} e^{itT(k)}$$

then $$(\Omega_+(k))_* = \operatorname{st}\lim_{t\to\infty} e^{-itT}(\tau^k_{-t})_* ,$$

and $$(\Omega_+(k)) = \sigma\text{-weak}\lim_{t\to\infty} \tau^k_{-t}\ \sigma_t .$$

(6.5) Each map τ^k_t is a *-map for k real in D, $t \in \mathbb{R}$.

(6.6) We can regard the infinitesimal generator of $(\tau^k_t)_*$ as representing $i\{T + k(\alpha_*\beta_* - \bar{\alpha}_*\bar{\beta}_*)\}$. This can be formally justified as in 5.11.

We now consider the automorphism property.

HYPOTHESIS 6.7.

Suppose $\nu \in M_*$ *has the following properties:*

i) ν *and* $\nu^* \in D_{\beta_*}$

ii) *There exist* ξ_i, η_i, f_i, g_i *in* $\mathcal{H}$ *satisfying*

$$\langle m, \beta_*\nu\rangle = \sum \langle m\,\xi_i, \eta_i\rangle$$

$$\langle m, \beta_*\nu^*\rangle = \sum \langle m\, f_i, g_i\rangle \qquad m \in M,$$

where $\sum\|\xi_i\|\cdot\|\eta_i\| < \infty$, $\sum\|f_i\|\cdot\|g_i\| < \infty$.

iii) $\xi_i, f_i \in D_A$ *and* $\sum\|A\xi_i\|\cdot\|\eta_i\|$, $\sum\|Af_i\|\cdot\|g_i\| < \infty$.

Then we say α, β *satisfy hypothesis* 6.7, *if for all such* ν,

$$\sum\langle A\,\xi_i, \eta_i\rangle = \sum\langle g_i, Af_i\rangle .$$

THEOREM 6.8.

Under the conditions of Theorem 6.2 *and Hypothesis* 6.7, τ^k_t *is a group of* *automorphisms for all* κ *in* D, t *in* $\mathbb{R}$.

PROOF.

It is enough to show that $\rho_t^k(1)\ \sigma_t^k(1) = 1$, for all t in $\mathbb{R}$, k in D.

This is achieved by showing that

$$\sigma_{-t}[\sigma_t^{\bar{k}}(1)^*] = \sigma_{-t}^k(1) \qquad t \in \mathbb{R},\quad k \in D;$$

i.e. we will inductively show that

$$\sigma_{-t}[\hat{U}_m(t)(1)^*] = U_m(t)(1) \qquad \text{for } m \geq o. \tag{6.9}$$

Let w denote the vector state $\xi \otimes \bar{\eta}$, where $\xi, \eta \in \mathcal{H}$.

Then
$$\langle U_m(t)(1)\xi,\eta\rangle = \langle S_{m-1}(1),\ Z_t(w)\rangle$$
$$= \lim_{n\to\infty} \int_o^t \langle (P_nA)\ U_{m-1}(s)(1),\ \beta_*\ e^{-iT(t-s)}w\rangle\ ds$$

and
$$\langle \sigma_{-t}\hat{U}_m(t)(1)\eta,\xi\rangle = \langle \hat{S}_{m-1}(1),\ \hat{Z}_t[(\sigma_{-t})_*w^*]\rangle$$
$$= \langle \hat{S}_o(1),\ (C_*)^{m-1}\ Z_t[(\sigma_{-t})_*w^*]\rangle$$
$$= \lim_{n\to\infty} \int_o^t \langle (P_nA)\ \hat{U}_o(s)1,\ \beta_*\hat{U}_{m-1}(t-s)_*(\sigma_{-t})_*w^*\rangle\ ds.$$

Thus 6.9 follows by induction and our hypothesis.

Hence

$$\rho_t^k(1)\ \sigma_t^k(1) = [\sigma_t^{\bar{k}}(1)]\ \sigma_t^k(1) = \sigma_t[\sigma_{-t}^k(1)].\ \sigma_t^k(1)$$
$$= \sigma_t\{\sigma_t^k(1)\ \sigma_{-t}\sigma_t^k(1)\} = \sigma_t\{\sigma_{-t}^k[1.\sigma_t^k(1)]\}$$
$$= \sigma_t(1) = 1.$$

The above theorem can be regarded as the algebra level analogue of [4, (4.1)].

REMARK 6.10.

It is possible to treat potentials of the form $\sum \beta_n\ \alpha_n$ using our method, if we replace $\mathcal{G}_M(\mathbb{R}^+)$ by $\mathcal{D}_M(\mathbb{R}^+ \times \mathbb{N})$, the natural subspace in $B(\mathcal{H},\ L^q(\mathbb{R}^+ \times \mathbb{N};\ \mathcal{H}))$.

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