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Minimal Algebras for Relativistic Wave Equations

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Abstract

The idea that matrices occuring in both first and second order relativistic wave equations generate (under commutation) some finite Lie algebra, which contains the Lorentz algebra, is considered. For first and second order wave equations the minimal non trivial Lie algebras are $so(3,2)$ and $sl(4,\mathbb{R})$ respectively.

The unique mass condition and the $so(3,2)$ algebra rule out all but the Dirac and Duffin Kemmer equations, while the $sl(4,\mathbb{R})$ algebra is associated to the Klein Gordan, Proca and Joos Weinberg (spin 1) equations.

1. Introduction

The question of constructing single mass relativistic wave equations for arbitrary spin, s, is essentially a straightforward matter. Since a unique mass is guaranteed by the Klein-Gordon equation we need only select a representation, $\mathcal{D}$, of SL(2, C) for the wave function, make sure that the particular spin, s, which we want occurs at least once in the reduction of $\mathcal{D}$ with respect to SU(2), and then project out this spin by applying to the wave function a suitable projection operator [1]. To describe a single particle of mass, m, and spin, s, we must ascertain that our projection operator projects onto a single irreducible SU(2) representation (with Casimir invariant s(s + 1)). It is for this reason that our projector operator is different from $W_\mu W^\mu - s(s+1) P_\mu P^\mu$, where W_μ is the Pauli Lubanski vector, except in the particular case where s occurs exactly once in the reduction of $\mathcal{D}$. The wave equation constructed using this projection operator is just a covariant statement of the fact that, in the rest frame, the wave function has exactly 2s + 1 independent components.

At any rate the very existence of this projection operator is the price we pay for having covariant wave functions. The reason we use covariant wave functions rather than, say, Wigner wave functions [2] is that they facilitate the introduction of Lorentz covariant local interactions. Well known examples of this are the electromagnetic and Yang-Mills [3] interactions which are usually introduced by way of a "minimal" principle. Since our initial free particle is described by two equations, namely the Klein-Gordon equation and the projection operator equation, we have the problem of introducting interactions into this system of equations in a consistent manner. Although, as is well known, there is no general solution to this problem, one would at least wish to avoid situations which would "obviously" forbid non-trivial interactions [4]. One possibility in this respect is to drop the Klein-Gordon equation completely and introduce interactions directly into the projection operator wave equation. This suggestion, however, has its own troubles as can be seen from an examination of the Joos-Weinberg spin $\frac{3}{2}$ equation in an external electromagnetic field [5]. A second, more aesthetically pleasing, alternative is to first combine the Klein-Gordon equation and the projector operator equation into a single wave equation and then introduce interactions directly into this new equation. Although this latter alternative still has its own problems [6], it is the one which is most generally accepted and it is this point of view which we shall take here. Namely, we shall consider a particle of mass m and spin s which is described by a single wave equation in which all components are initially independent. The only equations of this type known to us are for spins $\leqslant 3$ [7] and, although there is a general recipe known for constructing such equations [8], the explicit construction is an extremely laborious business as can be seen from, say, the spin 2 case. Not all the aforementioned equations are first order equations and although they may all be linearized in a straightforward manner the number of superfluous components in the wave function increases considerably. Also, not surprisingly, little work has been done on many of these equations, especially spins $\frac{5}{2}$, 3 and, to a lesser extent, spin 2 as far as either external interactions or the properties of the matrices occurring in these equations are concerned [9].

What we wish to examine here is some properties of the matrices occurring in these equations. More precisely we wish to consider the effect of assuming that these matrices generate (under commutation) some finite Lie algebra which contains the homogeneous Lorentz algebra. This idea is not new and is in fact implicit in the early work of Bhabha. The hope here was, presumably, that all the then known wave equations could be grouped in a simple manner according to the various Lie algebrae which the matrices occurring in these equations closed. A killer blow was seemingly dealt to this hope by Gupta [10] who found that the expression for the Lorentz generators in terms of the wave equation matrices for the Fiertz-Pauli-Gupta spin $\frac{3}{2}$ field was exceedingly complicated and certainly did not correspond to a simple Lie algebra. We take up this idea again here and shall examine, for both first and

second order wave equations, the simplest non-trivial Lie algebra which can be generated by the wave equation matrices. In all these cases the wave function space carries an irreducible representation of the Lie algebra in question. Equations of this type have been called irreducible wave equations by Castell [11].

For first order wave equations we are led to the Lie algebra $so(3,2)$ which, together with the unique mass condition, rules out all but the Dirac and Duffin-Kemmer wave equations. The relation of the Dirac and Duffin-Kemmer wave equations to $\overline{SO(3,2)}$ is of course well known. As a result there are few new results here. On the other hand, previously known results are derived in a general and, hopefully, more transparent manner. For second order wave equations we find that the well-known spin 0 and spin 1 equations also have an associated Lie algebra. This time it is $sl(4,\mathbb{R})$ and we show how the Klein-Gordon equation, the Proca equation and the Joos-Weinberg [12] spin 1 equation all fit neatly into irreducible representations of this algebra.

We use the metric $g_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$ and units where $\hbar = c = 1$.

2. General Formalism.

Consider the general first order linear equation of the form

$$(i\beta_\mu\partial^\mu - m)\,\psi(x) = 0 \tag{2.1}$$

where the β_μ are four finite dimensional matrices and ψ is a column vector. Under Poincaré transformations of the form

$$x' = \Lambda x + a$$

ψ is assumed to transform according as

$$\psi(x) \longrightarrow \psi'(x') = S(\Lambda)\,\psi(x)$$

Here $S(\Lambda)$ is a representation of the homogeneous Lorentz group in the space of the wave function ψ. The representation $S(\Lambda)$ has generators $S_{\mu\nu}$ which satisfy

$$i\left[S_{\mu\nu}, S_{\rho\sigma}\right] = g_{\mu\rho} S_{\nu\sigma} - g_{\nu\rho} S_{\mu\sigma} - g_{\mu\sigma} S_{\nu\rho} + g_{\nu\sigma} S_{\mu\rho} \tag{2.2}$$

The invariance of equation (2.1) under Poincaré transformations (i.e. so that also satisfies (2.1)) requires that the matrices β_μ satisfy

$$S^{-1}(\Lambda)\, \beta_\mu\, S(\Lambda) = \Lambda_\mu{}^\nu \beta_\nu \tag{2.3}$$

for every Lorentz transformation Λ. The condition in turn puts severe restrictions on the universal enveloping algebra $\mathcal{U}(\beta)$ of the β-matrices, i.e. the algebra consisting of all quantities of the form

$$C = c_\mu \beta^\mu + c_{\mu\nu}\beta^\mu\beta^\nu + c_{\mu\nu\rho}\beta^\mu\beta^\nu\beta^\rho + \cdots$$

where $c_{\mu\ldots}$ are complex numbers. In particular Harish-Chandra has shown [13] that if the β-matrices form an irreducible set, and we shall assume here that they do, then $\mathcal{U}(\beta)$ is finite, simple and its defining relations may be put in the form of Lorentz tensor equations.

Next we assume that the wave equation (2.1) describes a particle with unique mass $m > 0$. It can be shown [13, 14] that sufficient and, under very general assumptions, necessary conditions for this to hold are that the β-matrices should satisfy

$$\Sigma\, Q\, (g_{\mu_1\mu_2} - \beta_{\mu_1}\beta_{\mu_2})\,\beta_{\mu_3}\cdots\beta_{\mu_n} = 0 \tag{2.4}$$

where n is some integer ≥ 2 and Q denotes any permutation of the indices $\mu_1 \ldots \mu_n$. For $n \geq 3$ the abstract algebra defined by equation (2.4) is not finite. Thus the

β-matrices must satisfy some other stronger Lorentz tensor condition compatible with equation (2.4) in order to make the algebra finite. Let us suppose also that β_μ, or some scalar multiples thereof, generate (under commutation) a finite dimensional Lie algebra $\mathcal{L}(\beta)$, [which contains the homogeneous Lorentz algebra $so(3, 1)$], and consider the abstract algebra of the β-matrices subject to this condition and to equation (2.4). In an irreducible representation of this algebra anything which commutes with all the β-matrices is a multiple of the identity. The Casimir invariants of $\mathcal{L}(\beta)$ commute with all the β-matrices. Thus the representation of the β-matrices in question is also an irreducible representation of the Lie algebra $\mathcal{L}(\beta)$. For a large number of choices of $\mathcal{L}(\beta)$ all finite dimensional representations are known. Thus, for a given Lie algebra $\mathcal{L}(\beta)$, to find out which wave equations are allowed we simply have to select out (by hook or by crook) those irreducible representations of $\mathcal{L}(\beta)$ which have the property that β_0 acts as a projection operator in the sense that $\beta_0^n = \beta_0^{n-2}$. This last condition is just another version of equation (2.4).

In the next section we shall examine the choice $\mathcal{L}(\beta) = so(3,2)$ and show that this condition, when combined with equation (2.4) forces the abstract algebra of the β-matrices to be finite. There we shall make use of the infinitesimal version of equation (2.3). It is

$$i\left[\beta_\mu, S_{\kappa\rho}\right] = g_{\mu\rho}\beta_\kappa - g_{\mu\kappa}\beta_\rho \tag{2.5}$$

where $S_{\kappa\beta}$ are the generators of the representation $S(\Lambda)$.

3. First order equations.

We shall now consider a couple of simple choices for the Lie algebra $\mathcal{L}(\beta)$. The simplest choice is clearly the case where

$$i[\beta_\mu, \beta_\nu] = 0 \tag{3.1}$$

This case, however, is known to lead to nilpotent matrices β_μ which are only consistent with m = 0 so we shall rule it out. The next simplest possibility is that

$$i\kappa[\beta_\mu, \beta_\nu] = S_{\mu\nu} \tag{3.2}$$

where κ is some non-zero constant [15]. Now $\mathcal{L}(\beta)$ consists of the homogeneous Lorentz algebra together with a vector β_μ which has been added in a minimal way. $\mathcal{L}(\beta)$ is thus either $so(4, 1)$ or $so(3, 2)$. The possibility of $so(4, 1)$ is ruled out since in this case β_0 corresponds to a noncompact generator and is consistent only with m pure imaginary. We are thus led naturally to the case of $so(3, 2)$. Here β_0 corresponds to a compact generator so there is a representation of the β-matrices in which β_0 is hermitian. Setting all indices equal to 0 in equation (2.4) we find that the only eigenvalues of β_0 are 0 and ± 1. Thus equation (2.8) holds with n = 3. In particular we have

$$\beta_\mu^3 = g_{\mu\mu}\beta_\mu \tag{3.3}$$

for any $\mu = 0, 1, 2, 3$. Combining equations (2.5) and (3.2) we have

$$\beta_\mu\beta_\alpha\beta_\beta - \beta_\mu\beta_\beta\beta_\alpha - \beta_\alpha\beta_\beta\beta_\mu + \beta_\beta\beta_\alpha\beta_\mu = \frac{1}{\kappa} g_{\mu\alpha}\beta_\beta - \frac{1}{\kappa} g_{\mu\beta}\beta_\alpha \tag{3.4}$$

Setting $\mu = \alpha$ and $\beta \neq \mu$ this gives (no summation)

$$\beta_\mu^2\beta_\beta - 2\beta_\mu\beta_\beta\beta_\mu + \beta_\beta\beta_\mu^2 = \frac{1}{\kappa} g_{\mu\mu}\beta_\beta \tag{3.5}$$

Multiplying this equation on the right and on the left by β_μ and using equation (3.3) we find that

$$\left.\begin{aligned} g_{\mu\mu}\left(2-\frac{1}{\kappa}\right)\beta_\mu\beta_\beta\beta_\mu &= 2\beta_\mu^2\beta_\beta\beta_\mu^2 \\ \left[\left(2-\frac{1}{\kappa}\right)^2-4\right]\beta_\mu\beta_\beta\beta_\mu &= 0 \end{aligned}\right\} \tag{3.6}$$

Thus $\kappa = \frac{1}{4}$ or $\kappa \neq \frac{1}{4}$ and

$$\beta_\mu\beta_\beta\beta_\mu = 0 \tag{3.7}$$

for $\mu \neq \beta$. Consider first the case where $\kappa \neq \frac{1}{4}$ and equation (3.7) holds. Multiplying equation (3.5) by β_β and using (3.7) we obtain

$$\kappa = 1.$$

When μ, α and β are all different equation (3.4) may be written as

$$\beta_\mu\beta_\alpha\beta_\beta + \beta_\beta\beta_\alpha\beta_\mu = \beta_\mu\beta_\beta\beta_\alpha + \beta_\alpha\beta_\beta\beta_\mu$$

Multiplying this equation on the right by β_α^2 and using (3.5) we find easily that

$$\beta_\mu\beta_\beta\beta_\alpha + \beta_\alpha\beta_\beta\beta_\mu = 0$$

Thus $\beta_\mu\beta_\beta\beta_\alpha + \beta_\alpha\beta_\beta\beta_\mu$ is symmetric in α and μ and vanishes when μ, α and β are all different so there exist quantities α_λ, α'_λ such that

$$\beta_\mu\beta_\beta\beta_\alpha + \beta_\alpha\beta_\beta\beta_\mu = g_{\mu\alpha}\alpha'_\beta + g_{\mu\beta}\alpha_\alpha + g_{\beta\alpha}\alpha_\mu \tag{3.8}$$

Setting $\mu = \alpha \neq \beta$ and using (3.7) we obtain $\alpha'_\beta = 0$. Finally $\mu = \alpha = \beta$ gives $\alpha_\mu = \beta_\mu$. Thus equation (3.8) becomes

$$\beta_\mu\beta_\beta\beta_\alpha + \beta_\alpha\beta_\beta\beta_\mu = g_{\mu\beta}\beta_\alpha + g_{\mu\alpha}\beta_\beta \tag{3.9}$$

which are easily recognized as the defining relations of the Duffin Kemmer algebra [9]. This algebra is semi-simple and has 126 elements. Its irreducible representations have dimensions 1, 5 and 10 respectively. The 1 dimensional representation is trivial ($\beta_\mu = 0$) while the 5 and 10 dimensional representations describe particles with unique spins 0 and 1 respectively.

Let us now turn to the case $\kappa = \frac{1}{4}$. Equation (3.5) now gives, for $\mu \neq \beta$

$$\beta_\mu^2 \beta_\beta - 2\beta_\mu\beta_\beta\beta_\mu + \beta_\beta\beta_\mu^2 = 4 g_{\mu\mu}\beta_\beta$$

Multiplying this equation to the right and to the left by β_μ^2 and using (3.6) with $\kappa = \frac{1}{4}$ it is easy to derive

$$\begin{aligned} \beta_\beta\beta_\mu^2 &= -\beta_\mu\beta_\beta\beta_\mu \\ \beta_\mu^2\beta_\beta &= -\beta_\mu\beta_\beta\beta_\mu . \end{aligned} \qquad (3.10)$$

It follows easily that, for all μ, β we have

$$[\beta_\beta, \beta_\mu^2] = 0$$

Thus, since we have assumed that the β-matrices are irreducible, each β_μ^2 is a multiple of the identity. This multiple is fixed to be one by equation (3.3). Combining this with equation (3.10) we have

$$[\beta_\mu, \beta_\beta]_+ = 2 g_{\mu\beta} \qquad (3.11)$$

which are of course the defining relations of the Dirac algebra. This algebra is simple and consequently has only one irreducible representation which is 4 dimensional. This representation describes a particle with spin $\frac{1}{2}$.

Although the relationship between the Dirac and Duffin Kemmer equations to the Lie algebra ~~so~~(3, 2) has been discussed at length in the literature we shall summarize briefly for the reader's convenience some of these properties. Despite

the fact that the group $\overline{SO(3, 2)}$ is noncompact all its finite dimensional representations can be specified by the values of the two Casimir invariants [16]

$$\begin{aligned} I_2 &= S_{\mu\nu}S^{\mu\nu} + 2V_\mu V^\mu \\ I_4 &= \frac{1}{16}\left(S^D_{\mu\nu}S^{\mu\nu}\right)^2 + S^D_{\mu\alpha}V^\alpha S^{D\mu\beta}V_\beta \end{aligned} \qquad (3.12)$$

where $S_{\mu\nu} = i[V_\mu, V_\nu]$ and $S^D_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}S^{\alpha\beta}$. The required $\overline{SO(3, 2)}$ properties may be read directly from the table.

	Dirac	Duffin-Kemmer	
Representation	$(\frac{1}{2},0)+(0,\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})+(0,0)$	$(1,0)+(\frac{1}{2},\frac{1}{2})+(0,1)$
Dimension of representation	4	5	10
V_μ	$\frac{1}{2}\beta_\mu$	β_μ	β_μ
I_2	5	8	12
I_4	$\frac{45}{16}$	0	12

The $\overline{SO(3, 2)}$ representations are given in terms of their irreducible Lorentz components.

As already mentioned in section 2, we have made the (strong) assumption that the β-matrices formed an irreducible set. The manipulations described here are novel in that they demonstrate how this assumption works in practice. Despite this fact, all the main results of this section were known to Harish Chandra [17] and to Bhabba [15], although no proofs were presented.

4. Second order equations.

In contrast to the previous section our results for second order wave equations will be somewhat incomplete. This is partly due to the fact that, while the matrix algebras which turned up in our analysis of first order equations were well known, hardly any work at all of this nature has been done for second order wave equations. Thus we have little to fall back on here. Nevertheless we shall discuss as best we can the problems which arise in this case and, hopefully, make some inroads into their solutions.

Consider then the simplest possible type of second order wave equation, namely, one containing no term linear in ∂_μ. This is

$$(\Lambda_{\mu\nu}\partial^\mu\partial^\nu + m^2)\,\psi = 0 \qquad (4.1)$$

where $\Lambda_{\mu\nu}$ are 16 finite dimension matrices which we may take, without any loss of generality, to be symmetric in μ and ν. The matrices $\Lambda_{\mu\nu}$ may be analysed in a manner similar to that used to discuss the β-matrices in section 2. In this respect we shall only quote, without proof, those results which we need. The Poincaré invariance of equation (4.1) is guaranteed by the requirement that the Λ-matrices satisfy

$$i[\Lambda_{\mu\nu}, S_{\kappa\beta}] = g_{\mu\beta}\Lambda_{\kappa\nu} - g_{\mu\kappa}\Lambda_{\beta\nu} + g_{\nu\beta}\Lambda_{\mu\kappa} - g_{\nu\kappa}\Lambda_{\mu\beta} \qquad (4.2)$$

where $S_{\mu\nu}$ are the generators of homogeneous Lorentz transformations in the space of the wave function ψ. As in section 2 we shall assume that the matrices Λ generate a Lie algebra $\mathcal{L}(\Lambda)$ under commutation. Clearly $\mathcal{L}(\Lambda)$ must contain the homogeneous Lorentz algebra and a symmetric second rank tensor. If we wish to choose $\mathcal{L}(\Lambda)$ in a minimal manner as in section 3 we are led naturally to the Lie algebra $\mathfrak{sl}(4, \mathbb{R})$. Thus we have

$$i\kappa[\Lambda_{\mu\nu}, \Lambda_{\rho\sigma}] = g_{\mu\sigma}S_{\nu\rho} + g_{\mu\rho}S_{\nu\sigma} + g_{\nu\sigma}S_{\mu\rho} + g_{\nu\rho}S_{\mu\sigma} \qquad (4.3)$$

where κ is some non-zero constant.

Using the fact that Λ_{00} corresponds to a compact operator in $\mathfrak{sl}(4, \mathbb{R})$ it can be shown, in a way similar to that used in section 3, that the unique mass condition here becomes [18]

$$\begin{aligned}&[\Lambda_{\mu\nu}, \Lambda_{\kappa\beta}]_+ + [\Lambda_{\mu\kappa}, \Lambda_{\nu\beta}]_+ + [\Lambda_{\mu\beta}, \Lambda_{\nu\kappa}] \\ &= g_{\mu\nu}\Lambda_{\kappa\beta} + g_{\kappa\beta}\Lambda_{\mu\nu} + g_{\mu\kappa}\Lambda_{\nu\beta} + g_{\nu\beta}\Lambda_{\mu\kappa} + g_{\mu\beta}\Lambda_{\nu\kappa} + g_{\nu\kappa}\Lambda_{\mu\beta}.\end{aligned} \qquad (4.4)$$

What we would hope for is that the equations (4.2), (4.3) and (4.4) force κ to take only a few distinct values and that the algebra generated by these equations is finite. At any rate two particular irreducible representations of this algebra are given by

$$\Lambda_{\mu\nu} = g_{\mu\nu} \qquad (4.5)$$

which just gives us the Klein Gordon equation for spin 0, and

$$(\Lambda_{\mu\nu})_{\kappa\beta} = g_{\mu\nu}g_{\kappa\beta} - \tfrac{1}{2}(g_{\mu\kappa}g_{\nu\beta} + g_{\mu\beta}g_{\nu\kappa}) \qquad (4.6)$$

which leads to the Proca equation,

$$[(\partial_\mu\partial^\mu + m^2)g_{\alpha\beta} - \partial_\alpha\partial_\beta]\,\psi^\beta = 0 \qquad (4.7)$$

for spin 1. This latter case corresponds to $\kappa = 4$ in equation (4.3).

Besides these two well known equations there is another wave equation which fits neatly into this framework and which, perhaps, is not as well known. This is the Joos-Weinberg spin 1 equation

$$(\gamma_{\mu\nu}\partial^\mu\partial^\nu - m^2)\,\psi = 0 \qquad (4.8)$$

where $\gamma_{\mu\nu}$ are the generalized Dirac matrices which satisfy

$$\sum Q\, \gamma_{\mu_1\mu_2}(g_{\mu_3\mu_4} - \gamma_{\mu_3\mu_4}) = 0$$

where Q denotes any permutation of the indices $\mu_1, \mu_2, \mu_3, \mu_4$. The representation space of the wave function ψ is $(1,0) + (0,1)$ so the $\gamma_{\mu\nu}$ are 6 x 6 matrices. As it stands equation (4.8) does not describe a particle with unique mass m. In fact it describes a particle with mass m or im. It is easy however to modify [19] equation (4.8) so that it does describe a particle with unique mass m. The modified version of (4.8) is just equation (4.1) with

$$\Lambda_{\mu\nu} = \frac{1}{2}\left(g_{\mu\nu} - \gamma_{\mu\nu}\right) \qquad (4.9)$$

Using the explicit representation of the γ-matrices given in ref. [12] it is quite straightforward to verify that $\Lambda_{\mu\nu}$ as given by equation (4.9) satisfies both equations (4.3) and (4.4) with $\kappa = 1$. Thus the modified Joos-Weinberg spin 1 equation is also accommodated by an irreducible representation of the algebra defined by equations (4.2), (4.3) and (4.4). We do not know whether or not these three representations we have found are in fact all the allowed irreducible representations of this algebra. What is needed is a detailed analysis of the algebra in question.

5. Conclusions and Summary.

We have investigated in this paper, for both first and second order wave equations, the possibility that the matrices which occur in these equations generate under commutation a finite Lie algebra and in each case we examined the smallest non-trivial Lie algebra possible. For first order wave equations the Lie algebra in question turned out to be $so(3, 2)$. This together with the unique mass assumption was sufficient to rule out all but the Dirac and Duffin-Kemmer equations. It is interesting to note here that both of these equations describe particles with a single spin, although no unique single spin assumption was made. [21].

For second order wave equations the Lie algebra to which we were led was $sl(4, \mathbb{R})$ and we found that different irreducible representations of it accommodated the Klein Gordon equation, the Proca equation and the Joos-Weinberg spin 1 equation. Again all wave equations which we examined described particles with single spins ≤ 1. Much work remains to be done on the algebra describing these equations. Finally we note that the usual spin 2 equation [20] is not a $sl(4, \mathbb{R})$ type equation since in this case it is easy to verify that $\Lambda_{oo}^2 \neq \Lambda_{oo}$ which is incompatible with equation (4.4).

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