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A HOLLOW WATER-BAG

J. L. Synge

Dublin Institute for Advanced Studies

Abstract: Bouvier and Janin have described stationary water-bag models of globular clusters, the defining condition being essentially that the total energy of a particle (per unit mass) is bounded above. In the present paper this demand is supplemented by the condition that the angular momentum of a particle (per unit mass) is bounded below. The resulting model is a hollow spherical shell for which the density vanishes on both inner and outer surfaces, but the particle-velocity vanishes on neither.

1. Introduction.

This paper is concerned with stationary water-bag models of globular clusters. The distribution function f is a function of three variables (r, v, w), where r is distance from the centre, v the magnitude of the velocity, and w the cosine of the angle between the vectors of position and velocity, so that the ranges are

$$r \geqslant 0, \quad v \geqslant 0, \quad -1 \leqslant w \leqslant 1. \tag{1.1}$$

In this reduced phase-space of three dimensions the element of volume is $8\pi^2 r^2 v^2$ dr dv dw. Let us normalise f so that the mass inside a sphere of radius r, with centre at the origin, is

$$M(r) = 8\pi^2 \int_{\xi=0}^{r} \iint f(\xi, v, w)\, \xi^2 v^2 \,dr\, dv\, dw. \tag{1.2}$$

The density is then

$$\rho(r) = (4\pi r^2)^{-1}\, dM/dr = 2\pi \iint f(r, v, w)\, v^2\, dv\, dw. \tag{1.3}$$

It is convenient to define the gravitational potential as

$$\phi(r) = G\int_0^r M(\xi)\, \xi^{-2}\, d\xi \quad , \tag{1.4}$$

vanishing at r = 0 and increasing steadily with r. It satisfies Poisson's equation

$$r^{-2} \frac{d}{dr}(r^2\, d\phi/dr) = 4\pi G\rho \quad . \tag{1.5}$$

The total energy of a star and its angular momentum (both per unit mass) are respectively

$$H = \tfrac{1}{2}v^2 + \phi, \qquad J = rv(1 - w^2)^{\frac{1}{2}}; \tag{1.6}$$

both of these are conserved as the star moves.

In a stationary water-bag model we put $f = \eta$, a constant, in some fixed domain D of the reduced phase-space, with f = 0 outside D. The domain D must be bounded by a 2-space with equation of the form F(H, J) = 0.

2. The hollow model.

Let D be defined by

$$H \leqslant H_o, \qquad J \geqslant J_o, \tag{2.1}$$

where H_o and J_o are positive constants. If we represent the coordinates (r, v, w) as rectangular Cartesians, the surface $H = H_o$ is a cylinder with generators parallel to the w-axis. Until we know the potential ϕ, we can make only a qualitative sketch of this cylinder; its trace on the plane $w = 0$ is a falling curve, as indicated in Fig. 1. On the other

Fig. 1

hand, the surface $J = J_o$ is well-defined; its trace on $w = 0$ is the hyperbola $rv = J_o$.

The equations of the two traces are

$$v^2 = 2(H_o - \phi), \qquad v = J_o/r, \tag{2.2}$$

and so at any intersection r must satisfy

$$H_o - \phi - \tfrac{1}{2} J_o^2/r^2 = 0. \tag{2.3}$$

It is clear that, if J_o is small enough, a first intersection must exist, at $r = r_P$, say. It is by no means obvious that a second intersection ($r = r_Q$, as shown in Fig. 1) exists. Let us proceed without assuming its existence.

The domain $r < r_P$ is empty, and so, by (1.4), $\phi = 0$ at $r = r_P$. So, by (2.3),

$$H_o - \tfrac{1}{2} J_o^2/r_P^2 = 0, \qquad J_o = (2H_o)^{\frac{1}{2}} r_P . \tag{2.4}$$

We may then regard H_o and r_P as basic constants, with J_o given as above. We see a spherical shell with inner radius r_P, but with closure at r_Q still unestablished. This does not invalidate the argument, because we shall proceed with r increasing from r_P.

Let us write the closure equation (2.3) in the equivalent form

$$\chi(r) = 0, \tag{2.5}$$

where

$$\chi(r) = 1 - \phi/H_o - r_P^2/r^2. \tag{2.6}$$

By (1.3) the density is

$$\rho(r) = 2\pi\eta \iint v^2 \, dv \, dw, \tag{2.7}$$

where the limits for w are $\pm [1 - J_o^2/(r^2 v^2)]^{\frac{1}{2}}$ and those for v are J_o/r and $[2(H_o - \phi)]^{\frac{1}{2}}$. Hence

$$\rho(r) = A\eta H_o^{3/2}[\chi(r)]^{3/2}, \quad A = (4\pi/3)2^{3/2}. \tag{2.8}$$

We are to substitute this in Poisson's equation (1.5), changing the variables from (ϕ, r) to (χ, ξ) where χ is as in (2.6) and

$$\xi = Cr, \quad C^2 = 4\pi AG\eta H_o^{\frac{1}{2}}. \tag{2.9}$$

This gives the differential equation

$$\xi^{-2}\frac{d}{d\xi}(\xi^2 \, d\chi/d\xi) + \chi^{3/2} + 2\,\xi_P^2/\xi^4 = 0, \tag{2.10}$$

where $\xi_P = Cr_P$. The range is $\xi \geqslant \xi_P$ with the initial conditions

$$\chi = 0, \quad d\chi/d\xi = 2/\xi_P \text{ for } \xi = \xi_P. \tag{2.11}$$

At this point we may compare formulae with those of Bouvier and Janin (1970). In their water-bag model the domain D is bounded by $H = H_o$ and $J = 0$, and so we might expect the present formulae to agree with theirs if we simply put $J_o = 0$, or equivalently $r_P = 0$. Then indeed the differential equation (2.10) reduces to the Emden equation, as in their work, but the initial conditions (2.11) are quite different. The explanation is as follows. No matter how small we make J_o, but still positive, the present model has a hole at the centre, whereas the Bouvier–Janin model has no hole. A closer investigation for small values of r_P shows that, although the central hole persists, the density near it builds up quickly to the Bouvier–Janin density, so that the small hole at the centre produces only a local effect. Thus there is no disagreement between the two models in the limiting case of a small central hole.

3. The theorem of closure.

It remains to show that the model closes at r_Q. Equivalently, that χ as defined by (2.10) and (2.11), has a second zero.

Let us write $\xi_P = \underset{\sim}{a}$ for notational simplicity. We note that $\underline{a}$ occurs both in the equation and in the initial conditions. To remove it from the latter, and at the same time make the total possible range of independent variable finite, define $z = a/\xi$. Then the equation becomes

read (a^2/z^4)

$$d^2\chi/dz^2 + (a^2/z^4)\chi^{3/2} + 2 = 0 \tag{3.1}$$

for the range $z \leqslant 1$ with initial conditions

$$\chi = 0, \quad d\chi/dz = -2 \text{ for } z = 1. \tag{3.2}$$

It is clear from (3.1) that the graph of $\chi(z)$ has no inflection and is concave downward. With a prime to indicate d/dz, integration gives

$$\chi' = -2z + a^2\int_z^1 t^{-4}[\chi(t)]^{3/2}dt. \tag{3.3}$$

From the conditions at z = 1, there exist positive numbers (z_1, ϵ) with $z_1 < 1$ such that $\chi(z_1) = \epsilon$. Now suppose that χ has no maximum: in other words

$$\chi' < 0 \quad \text{for } 0 < z < 1. \tag{3.4}$$

Then $\chi > \epsilon$ for $0 < z < z_1$. Therefore by (3.3)

$$\begin{aligned} \chi' &> -2z + a^2\int_z^{z_1} t^{-4}[\chi(t)]^{3/2}\,dt \\ &> -2z + a^2\epsilon^{3/2}\int_z^{z_1} t^{-4}\,dt \\ &= \tfrac{1}{3}a^2\epsilon^{3/2}(z^{-3} - z_1^{-3}) - 2z. \end{aligned} \tag{3.5}$$

This takes positive values in the range $0 < z < 1$. Therefore (3.4) is false. Thus there exists z_0 $(0 < z_0 < 1)$ such that

$$\chi'_0 = 0, \quad \chi_0 > 0. \tag{3.6}$$

Now integrate (3.1) back from $z = z_0$:

$$\chi' = -2(z - z_0) + a^2\int_z^{z_0} t^{-4}[\chi(t)]^{3/2}\,dt, \tag{3.7}$$

and

$$\chi = \chi_0 - (z - z_0)^2 - a^2 \int_z^{z_0} t^{-4}(t - z)[\chi(t)]^{3/2}\, dt. \qquad (3.8)$$

Compare this function with the function $z\chi_0/z_0$, which represents the straight line drawn in the (z, χ) plane from the origin to the point (z_0, χ_0). Assume that the solution χ satisfies

$$\chi \geqslant z\chi_0/z_0 \quad \text{for } 0 \leqslant z \leqslant z_0. \qquad (3.9)$$

Then by (3.8)

$$\chi \leqslant \chi_0 - a^2(\chi_0/z_0)^{3/2} \int_z^{z_0} t^{-4}(t - z)t^{3/2}\, dt, \qquad (3.10)$$

or

$$\chi \leqslant \chi_0 - \frac{2}{3} a^2(\chi_0/z_0)^{3/2} z^{-\frac{1}{2}}[1 - (z/z_0)^{\frac{1}{2}}]^2[2 + (z/z_0)^{\frac{1}{2}}]. \qquad (3.11)$$

For z small enough, but finite, this expression becomes negative. Therefore (3.9) is false. The graph of the solution χ cuts the straight line $z\chi_0/z_0$, and, since this graph is concave downward, there exists a value of z $(0 < z < 1)$ such that $\chi = 0$.

This establishes the closure of the model for $z = z_Q$, say, and hence for $\xi_Q = \xi_P/z_Q$, and hence for $r_Q = r_P/z_Q$. The finiteness of χ' there follows from (3.3), and hence the finiteness of $d\phi/dr$ at $r = r_Q$.

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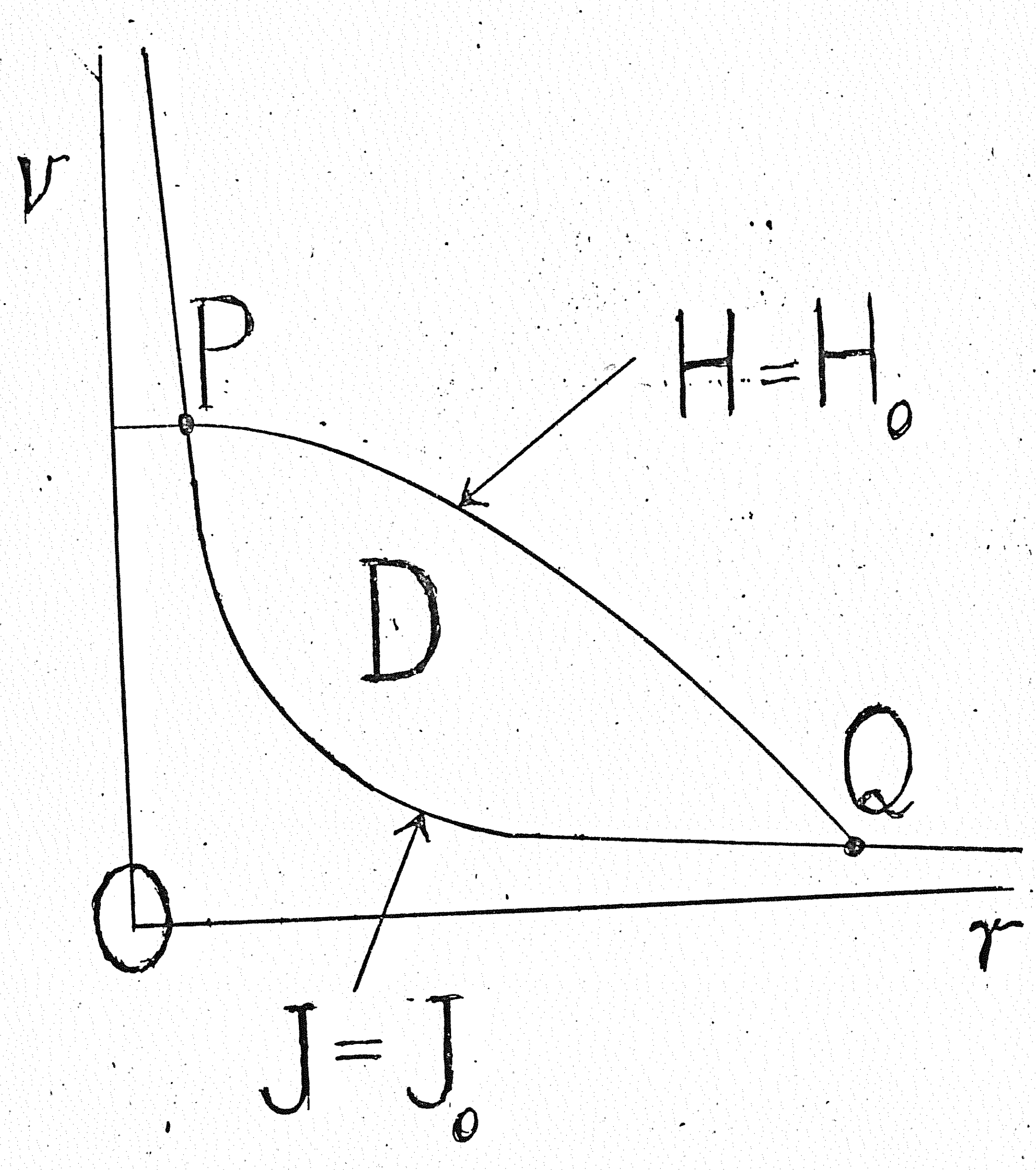

Fig 1